De Mathematics Competitions

1st Annual

DMC 10

Wednesday, September 30, 2020

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU DECIDE TO BEGIN.
2. This is a twenty-five question multiple choice test. For each question, only one answer choice is correct.
3. Mark your answer to each problem on the DMC 10 Answer Form with a keyboard. Check the keys for accuracy and erase errors and stray marks completely.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only blank scratch paper, rulers, and erasers are allowed as aids. Prohibited materials include calculators, smartwatches, phones, computing devices, compasses, protractors, and graph paper. No problems on the competition will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the competition, your non-existent proctor will not ask you to record certain information on the answer form.
8. You will have 75 minutes to complete the competition. You can discuss only with people that have already taken the competition in the private discussion forum until the end of the contest window.
9. When you finish the exam, don't sign your name in the space not provided on the Answer Form.

The DMC Committee reserves the right to disqualify scores from a school if it determines that the rules or the required security procedures were not followed.

The publication, reproduction or communication of the problems or solutions of this competition during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

Students who score well on this DMC 10 may or may not be invited to the 2021 DIME. More details about the DIME and other information are on the back page of this test booklet.

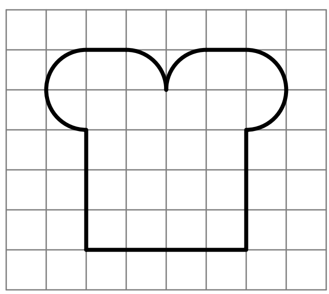
1. What is the value of
$$\frac{(2^0-2^1)^{2020}}{(2\cdot 0+2^0)^{2021}}?$$

(A) -1 **(B)** $-\frac{1}{2}$ **(C)** 0 **(D)** $\frac{1}{2}$ **(E)** 1

2. If the ratio of males to females in a country club is exactly 9 to 5, and there are fewer than 100 people in the club, what is the largest possible number of people in the club? (Assume that all of the people in the club are either male or female.)

(A) 95 **(B)** 96 **(C)** 97 **(D)** 98 **(E)** 99

3. The figure below has been drawn on 1 cm $\times$ 1 cm graph paper, where every square not completely covered or empty is filled with a quarter circle. What is the area of the figure in cm^2?



(A) $12+\frac{3\pi}{2}$ **(B)** $8+3\pi$ **(C)** $18+\pi$ **(D)** $12+3\pi$ **(E)** $18+\frac{3\pi}{2}$

4. A dog has four legs, and a dug has three legs. Janelle has a whole number of dogs and dugs as pets, and she has no other pets. If there are 61 legs across all of Janelle's pets, what is the smallest possible number of dugs that Janelle could have?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

5. Rohan wants to distribute 25 slices of pizza to n people such that each person gets an equal number of slices, except for one person who gets one more slice than each of the other people. If n is greater than 1, how many different integer values of n exist?

   (A) 2 **(B)** 5 **(C)** 7 **(D)** 8 **(E)** 9

6. Two distinct elements x and y are chosen from the set $\{1, 2, 3, 4\}$ at random. What is the probability that the line with slope $\frac{y}{x}$ passing through the point (x, y) also passes through the point $(2020, 1010)$?

   (A) $\dfrac{1}{12}$ **(B)** $\dfrac{1}{6}$ **(C)** $\dfrac{1}{4}$ **(D)** $\dfrac{1}{3}$ **(E)** $\dfrac{1}{2}$

7. Anthony, Daniel, and Richard have 17, 20, and 26 trading cards, respectively. Every minute, one of the three boys gives away two of his trading cards such that the other two boys get one trading card each. What is the shortest amount of time, in minutes, that it could take for the three boys to each have an equal number of trading cards?

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

8. Two distinct points A and B are chosen on the circumference of a circle with center O. Another point C, distinct from A and B, is chosen on the circumference. If $\angle AOB = 70^\circ$, what is the probability that $\triangle ABC$ is acute?

   (A) $\dfrac{7}{36}$ **(B)** $\dfrac{7}{18}$ **(C)** $\dfrac{1}{2}$ **(D)** $\dfrac{11}{18}$ **(E)** $\dfrac{31}{36}$

9. Alice and Bob are racing each other on a track. Each of their lanes are 400 meters in length. Normally, Alice and Bob run at constant rates of a and b meters per minute, respectively, but Alice's lane has a 180-meter sand region in the middle, in which she runs at three-quarters of her normal speed. If Alice and Bob take the same amount of time to run through their lanes without stopping, what is $\frac{a}{b}$?

   (A) 1.05 **(B)** 1.15 **(C)** 1.25 **(D)** 1.35 **(E)** 1.45

10. What is the largest integer n for which there exists an ordered triple (p, q, r) of distinct prime numbers such that $p^2(q^2 + r^2)$ is divisible by 2^n?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

11. Let $f(x) = x^2 - x - 3$ and $g(x) = 2x + 3$ for all real numbers x. What is the sum of all real values of x such that $f(g(x)) = g(f(x))$?

(A) -6 **(B)** $-\frac{9}{2}$ **(C)** -3 **(D)** $-\frac{3}{2}$ **(E)** 0

12. Let A and B be two distinct points on a plane. Let $\mathcal{S}$ denote the set of all circles on the plane with a finite area such that A and B are on the circumference of the circle. What is the region of all points not on the circumference of any of the circles in $\mathcal{S}$?

(A) Every point on line AB excluding A and B

(B) Every point on segment $\overline{AB}$ excluding A and B

(C) Every point on line AB but not on segment $\overline{AB}$

(D) The midpoint of segment $\overline{AB}$

(E) None of the above

13. 10 students are taking a final exam. Of the 10 students, 3 of them are guaranteed to pass. However, the other 7 students are lazy and are not guaranteed to pass, but each of them has the same probability of passing as one another, where the probability is nonzero. If Tomo is one of the 7 lazy students, and exactly 6 out of the 10 students passed the exam, what is the probability that Tomo was one of those 6 students?

(A) $\frac{5}{16}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{3}{7}$ **(E)** $\frac{5}{9}$

14. Let n be the 2020th smallest positive integer such that the sum of the digits in its base-nine representation is divisible by 8. What is the sum of the digits in the base-ten representation of n, in base-ten?

(A) 14 **(B)** 16 **(C)** 18 **(D)** 20 **(E)** 22

15. In triangle ABC, $AB = 1$, $BC = \sqrt{3}$, and $AC = 2$. Let D denote the midpoint of side $\overline{AC}$, and let O_1 and O_2 denote the centers of the circumcircles of $\triangle ABD$ and $\triangle DBC$, respectively. What is the area of $\triangle BO_1O_2$?

(A) $\frac{\sqrt{3}}{6}$ **(B)** $\frac{\sqrt{3}}{4}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{\sqrt{3}}{2}$ **(E)** $\sqrt{3}$

16. There exist three positive integers such that the sum of the integers is 14, the sum of the squares of the integers is 78, and the sum of the cubes of the integers is 476. The sum of the reciprocals of the integers can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

(A) 126 **(B)** 127 **(C)** 128 **(D)** 129 **(E)** 130

17. 8 people randomly split into 2 groups of four to dance. After that, the 8 people randomly split into 4 pairs of two to talk. What is the probability that exactly 2 of the 4 pairs contain two people who have danced in the same group of four?

(A) $\frac{8}{35}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{4}{21}$ **(D)** $\frac{24}{35}$ **(E)** $\frac{6}{7}$

18. A plane cuts into a sphere of radius 11 such that the area of the region of the plane inside the sphere is 108π. A perpendicular plane cuts into the sphere such that the area of the region of the plane inside the sphere is 94π. Given that the two planes intersect at a line, what is the length of the segment of the line inside the sphere?

(A) $6\sqrt{3}$ **(B)** 12 **(C)** $11\sqrt{2}$ **(D)** $8\sqrt{5}$ **(E)** 18

19. Let the sum of $n \geq 2$ consecutive integers be a positive prime number, where the smallest of the integers is a. If $a+n = 28$, what is the sum of all possible values of a?

(A) -26 **(B)** -25 **(C)** -1 **(D)** 0 **(E)** 1

20. In trapezoid $ABCD$ with $\overline{AD} \parallel \overline{BC}$ and side lengths $AD = 18$, $BC = 20$, and $AB = CD = 8$, let X be the intersection of line AB and the bisector of $\angle ADC$, and let Y be the intersection of line CD and the bisector of $\angle DAB$. What is XY?

(A) 22 **(B)** 24 **(C)** 25 **(D)** 27 **(E)** 28

21. A set of positive integers exists such that for any integer k in the set, all of the values k^2+2, k^2+4, and k^2+8 are prime numbers. Two distinct integers m and n are chosen from the set. Which of the following is a possible value of $m+n$?

(A) 40 **(B)** 56 **(C)** 72 **(D)** 88 **(E)** 104

22. Define a sequence recursively by $T_1 = 1$ and

$$T_n = \frac{n! \cdot T_{n-1}}{(n-1)! + n \cdot T_{n-1}}$$

for all integers $n \geq 2$. The value of T_{2020} can be written as $\frac{2020!}{m}$, where m is a positive integer. What is the sum of the distinct prime divisors of m?

(A) 198 **(B)** 199 **(C)** 200 **(D)** 201 **(E)** 202

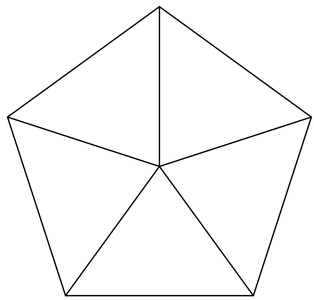
23. Joy picks an integer n from the interval $[1, 40]$. She tells Amy the remainder when n is divided by 7 and Sid the number of divisors of n. Amy and Sid both know n is in the interval $[1, 40]$, but they get confused and believe Amy was told the number of divisors and Sid was told the remainder. Amy says, "I know what n is." Sid replies, "If so, then I also know what n is." As it turns out, they thought of the same value but were wrong due to their confusion. If Amy and Sid tell the truth based on their beliefs and can reason perfectly, what is the sum of all possible actual values of n?

(A) 21 **(B)** 22 **(C)** 23 **(D)** 24 **(E)** 25

24. In triangle ABC, $AB = 16$ and $BC = 8$, with a right angle at C. Let M be the midpoint of side $\overline{AB}$, let N be a point on side $\overline{AC}$, and let P be the intersection of segments $\overline{BN}$ and $\overline{CM}$. If $BP = 7$, what is the sum of all possible values of $\frac{CN}{AN}$?

(A) $\frac{23}{21}$ **(B)** $\frac{21}{19}$ **(C)** $\frac{19}{17}$ **(D)** $\frac{17}{15}$ **(E)** $\frac{15}{13}$

25. The diagram below is composed of five triangles forming a regular pentagon. How many ways are there to color the 10 edges in the diagram with red, blue, or green in such a way that no vertex of a triangle has 3 or more edges connected to it that are the same color? Rotations and reflections are considered distinct colorings.



(A) 7290 **(B)** 9750 **(C)** 11,790 **(D)** 12,330 **(E)** 13,350

DMC 10

DO NOT OPEN UNTIL WEDNESDAY, September 30, 2020

****Administration on an earlier date will disqualify your results.****

- All the information needed to administer this exam is not contained in the non-existent DMC 10 Teacher's Manual. PLEASE READ THE MANUAL BEFORE WEDNESDAY, SEPTEMBER 30, 2020.
- Send **DeToasty3** and **nikenissan** a private message submitting your answers to the DMC 10. AoPS is the only way to submit your answers.
- The publication, reproduction or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.

For more information about the DMC and our other competitions, please visit `https://detoasty3.github.io/dmc.html`.

Questions and comments about this competition should be sent to:

DeToasty3.

The problems and solutions for this DMC 10 were prepared by the DMC Editorial Board under the direction of:

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