Official Solutions

De Mathematics Competitions

2nd Annual

DMC 10C

Friday, June 11, 2021

This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

Questions and complaints about this competition should be
sent by private message to

DeToasty3.

Answer Key:

1. (B)	2. (D)	3. (D)	4. (A)	5. (C)
6. (E)	7. (D)	8. (B)	9. (C)	10. (C)
11. (D)	12. (D)	13. (D)	14. (C)	15. (A)
16. (A)	17. (E)	18. (C)	19. (D)	20. (C)
21. (D)	22. (B)	23. (C)	24. (B)	25. (C)

Problem 1:

(DeToasty3) What is the value of

$$4^1 - 3^2 + 2^3 - 1^4?$$

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Answer (B):

The requested answer is

$$4^1 - 3^2 + 2^3 - 1^4 = 4 - 9 + 8 - 1 = \boxed{\text{(B) } 2}.$$

Problem 2:

(HrishiP) Let a, b, c, and d be real numbers that satisfy

$$a + 1 = b, \quad b + 2 = c, \quad c + 3 = d, \quad d + 4 = 21.$$

What is $a + b + c + d$?

(A) 51 **(B)** 52 **(C)** 53 **(D)** 54 **(E)** 55

Answer (D):

We obtain

$$\begin{aligned} d + 4 = 21 &\implies d = 17 \implies c + 3 = 17 \implies c = 14 \\ \implies b + 2 = 14 &\implies b = 12 \implies a + 1 = 12 \implies a = 11. \end{aligned}$$

Therefore, the requested answer is

$$a + b + c + d = 11 + 12 + 14 + 17 = \boxed{\text{(D) } 54}.$$

■

Problem 3:

(DeToasty3) Bill writes all odd perfect squares from 1 to 100, inclusive, and Jill writes all even perfect squares from 1 to 100, inclusive. Who writes more digits, and by how many?

(A) Bill, 1 **(B)** Bill, 2 **(C)** Jill, 1 **(D)** Jill, 2 **(E)** neither

Answer (D):

Note that Bill writes 9 and Jill writes 16, and Bill writes 81 and Jill writes 100. Since in all other pairs of two perfect squares of odd then even in that order, the number of digits is the same, we have that Jill writes 2 more digits than Bill, for a requested answer of $\boxed{\textbf{(D) } \text{Jill, } 2}$. ■

Problem 4:

(DeToasty3) Given a right triangle with legs of lengths 5 and 6, a square is drawn with one side as its hypotenuse such that the triangle is completely inside the square. What is the area of the region inside the square but outside the triangle?

(A) 46 **(B)** 47 **(C)** 48 **(D)** 49 **(E)** 50

Answer (A):

By the Pythagorean Theorem, we get that the side length of the square is equal to

$$\sqrt{5^2 + 6^2} = \sqrt{61},$$

so the area of the square is equal to 61. The area of the triangle is equal to

$$\frac{1}{2} \cdot 5 \cdot 6 = 15$$

since it is a right triangle. Therefore, the requested area is $61 - 15 = \boxed{\textbf{(A) } 46}$. ■

Problem 5:

(pog) A group of 200 people were invited to see a movie, where each person either had first row seats, second row seats, or third row seats. It is given that four-fifths of the people invited chose to watch the movie, one-ninth of the viewers were not invited, one-fifth of the viewers had first row seats, and 60 of the viewers had third row seats. What is the probability that a randomly selected viewer had second row seats?

(A) $\frac{3}{10}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{7}{15}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{4}{5}$

Answer (C):

From the condition that four-fifths of the people invited chose to watch the movie, we get that

$$\frac{4}{5} \cdot 200 = 160$$

of the people invited chose to watch the movie. Next, from the condition that one-ninth of the viewers were not invited, so eight-ninths of the viewers invited. Letting n be the total number of viewers, we get

$$\frac{8n}{9} = 160 \implies n = 180.$$

Finally, from the condition that one-fifth of the viewers had first row seats, we get that

$$\frac{1}{5} \cdot 180 = 36$$

viewers had first row seats. Since 60 of the viewers had third row seats, the total number of viewers with second row seats is $180 - 36 - 60 = 84$, so the probability is

$$\frac{84}{180} = \boxed{\textbf{(C)}\ \frac{7}{15}},$$

as requested. ■

Problem 6:

(DeToasty3) John is playing a game with 6 levels, each with 5 stages. After the third stage of each of the first five levels, John may choose whether or not to skip the remaining stages in the level and start at the first stage of the next level. If John finished the whole game, how many possible combinations of stages could John have played through?

(A) 5 **(B)** 10 **(C)** 16 **(D)** 30 **(E)** 32

Answer (E):

We see that for each of the first five levels, John may either choose to do the remaining two stages or choose not to do so, for two total possibilities. Thus, the requested answer is $2^5 = \boxed{\text{(E) } 32}$. ■

Problem 7:

(DeToasty3) What is the sum of all positive real numbers a such that the equation $x^2 + ax - 12 = 0$ has two distinct integer solutions x?

(A) 6 **(B)** 12 **(C)** 14 **(D)** 16 **(E)** 22

Answer (D):

Let m and n be the two integer solutions x to the equation $x^2 + ax - 12 = 0$. By Vieta's Formulas, we get that $mn = -12$ and $m + n = -a$. Now, we just have to find all integers m and n such that a is positive. We get that

$$(m, n) \in \{(1, -12), (2, -6), (3, -4)\},$$

which gives us $a \in \{11, 4, 1\}$. The requested answer is $11 + 4 + 1 = \boxed{\text{(D) } 16}$. ■

Problem 8:

(DeToasty3) Daniel has to walk one mile to complete his gym homework. He decides to split his path into quarters, where after each quarter, he randomly chooses to turn 90° clockwise or counterclockwise with equal probability. If Daniel walks in a straight line each quarter, what is the probability that he will end up where he started after walking the mile?

(A) $\frac{1}{8}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{5}{8}$ **(E)** $\frac{3}{4}$

Answer (B):

In total, there are 2 possibilities for each of the three instances where Daniel turns, for a total of $2^3 = 8$ possible paths. Of these, it is easy to see that the only paths where Daniel ends up where he started after walking the mile are where Daniel turns clockwise at all three turns or counterclockwise at all three turns. Thus, the requested probability is

$$\frac{2}{8} = \boxed{\text{(B) } \frac{1}{4}}.$$

■

Problem 9:

(DeToasty3) Let a and b be positive integers. If a is divisible by 2 but not 3, and b is divisible by 3 but not 2, what is the greatest possible three-digit value of $a + b$?

(A) 995 **(B)** 996 **(C)** 997 **(D)** 998 **(E)** 999

Answer (C):

We see that if $a+b = 999$, then since b and 999 are both divisible by 3, then a must also be divisible by 3, which is a contradiction. Similarly, if $a + b = 998$, then since a and 998 are both divisible by 2, then b must also be divisible by 2, which is a contradiction. However, we can have $a + b = 997$; one possible construction for this is $a = 4$ and $b = 993$, which works. Therefore, the requested answer is $\boxed{\text{(C) } 997}$. ■

Problem 10:

(DeToasty3) How many orderings of the six numbers 1, 1, 2, 2, 3, and 6 are there such that the sum of the first three numbers is twice the sum of the last three numbers?

(A) 9 **(B)** 18 **(C)** 27 **(D)** 36 **(E)** 72

Answer (C):

Upon inspection, we see that there are two possibilities. On the one hand, the first three numbers can be some permutation of $(2, 2, 6)$, and the last three numbers can be some permutation of $(1, 1, 3)$, which gives us $3 \cdot 3 = 9$ orderings. On the other hand, the first three numbers can be some permutation of $(1, 3, 6)$, and the last three numbers can be some permutation of $(1, 2, 2)$, which gives us $6 \cdot 3 = 18$ orderings. Adding, we get the requested answer of $9 + 18 = \boxed{\text{(C) } 27}$. ■

Problem 11:

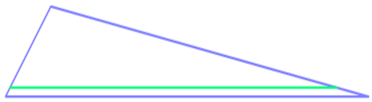
(richy) Given a triangle, a line is drawn such that it intersects the triangle in exactly two points. Which of the following statements must always be true?

(A) The triangle is split into a smaller triangle and quadrilateral.
(B) The segment of the line in the triangle is shorter than every side of the triangle.
(C) At least two of the triangle's angle bisectors meet the line inside the triangle.
(D) The perimeters of each of the regions formed are less than that of the triangle.
(E) None of the above.

Answer (D):

We see that choice **(A)** is not true because we may take a line that passes through one of the triangle's vertices and the opposite side.

We see that choice **(B)** is not true. Consider the diagram below for an example.



We see that choice **(C)** is not true. Consider the diagram below for an example.

We see that choice **(D)** is true. Let the length of the segment of the line in the triangle be L, and let the perimeters of the two regions formed be A and B. We see that the perimeter of the triangle is then $A + B - 2L$. However, by the Triangle Inequality, we see that $A - L > L$ and $B - L > L$, which means that $A - 2L > 0$ and $B - 2L > 0$. Therefore, we get that $A + B - 2L > A$ and $A + B - 2L > B$.

Thus, the requested answer is $\boxed{\text{(D)}}$. ■

Problem 12:

(DeToasty3) In a plane, eight rays emanate from a point P such that every two adjacent rays form an acute angle with measure $45°$. Next, a line segment with a finite length is drawn in the plane. If the line segment intersects exactly n of the rays, what is the sum of all possible values of n? (If the line segment passes through P, then $n = 8$.)

(A) 13 **(B)** 14 **(C)** 17 **(D)** 18 **(E)** 23

Answer (D):

We see that any $n \in \{1, 2, 3, 4\}$ is possible, and the constructions for each are left as an exercise to the reader. Additionally, $n = 8$ clearly works by letting the line segment pass through P. To show that n cannot equal 5, 6, or 7, we see that a line segment's angle is $180°$, while in order to pass through greater than 4 rays, it must have an angle greater than $180°$, which is a contradiction. Thus, the requested answer is $1+2+3+4+8 = \boxed{\textbf{(D) } 18}$. ■

Problem 13:

(HrishiP) A positive integer is called *fresh* if the sum of the squares of its digits in base-4 is greater than the integer itself. How many fresh positive integers are there?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Answer (D):

Say that such a positive integer has n digits in base-4. Then, we can establish the bound

$$n(3^2) > 4^{n-1}$$

or $n \leq 3$. Note that every 1 digit integer trivially works except 1, so there are 2 integers here. Then, if there are two digits, we have

$$4a + b \leq a^2 + b^2.$$

There are only 3 integers here. Finally, if there are three digits, we have

$$16a + 4b + c < a^2 + b^2 + c^2.$$

Note that for $a, b, c \in \{1, 2, 3\}$ we have $16a > a^2$ and $4b > b^2$, and the gap between $16a$ and a^2 is easily large enough to cover the gap between c and c^2. So, there no solutions here. Thus, there are a total of $\boxed{\textbf{(D) } 5}$ fresh positive integers, as requested. ■

Problem 14:

(DeToasty3) Draw two identical non-intersecting circles, a line tangent to both circles at distinct points A and B, where the circles are on the same side of the line, and a line tangent to both circles at distinct points C and D, where the circles are on opposite sides of the line. The lines intersect at point P. If $AB = 11$ and $CD = 5$, what is $AP \cdot BP$?

(A) 20 **(B)** 22 **(C)** 24 **(D)** 26 **(E)** 28

Answer (C):

Without loss of generality, let points A, P, and B be collinear in that order, and let points P, C, and D be collinear in that order. By equal tangents, we realize that $AP = CP$ and $BP = DP$. Let $AP = CP = x$ and $BP = DP = y$. Then, we have that $AB = AP + BP = x + y = 11$ and $CD = DP - CP = y - x = 5$. Solving, we obtain

$x = AP = 3$ and $y = BP = 8$. Thus, the requested answer is $AP \cdot BP = 3 \cdot 8 = \boxed{\textbf{(C) } 24}$.

Problem 15:

(pog) For certain real numbers x and y, the first 3 terms of a geometric progression are $x-2$, $2y$, and $x+2$ in that order, and the sum of these terms is 4. What is the fifth term?

(A) $\frac{64}{3}$ **(B)** $\frac{196}{9}$ **(C)** $\frac{512}{23}$ **(D)** $\frac{256}{11}$ **(E)** $\frac{128}{5}$

Answer (A):

We have that $(x-2)+2y+(x+2)=4$, so since

$$(x-2)+2y+(x+2)=2x+2y,$$

we get that $2x+2y=4$. Thus, $x+y=2$, so $y=2-x$. Consequently, the common ratio of the sequence is $\frac{2y}{x-2}=\frac{2(2-x)}{x-2}$. Since $2-x$ and $x-2$ cancel, this is equal to -2. Thus, $x+2=-2(2)(2-x)$, so solving gives $x=\frac{10}{3}$ and our answer is

$$\frac{4}{3}\cdot(-2)^4=\boxed{\textbf{(A) } \frac{64}{3}}.$$

Problem 16:

(HrishiP) In the coordinate plane, let $P=(10,0)$ and $Q=(a,a)$, for some real number a. It is given that when point P is reflected over point Q, the resulting point P' is 2 times as far from the origin as point P is. What is the sum of all possible values of a?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Answer (A):

First, to find the coordinates of P', which we will call (x,y), we must have

$$2a = x+10 \quad \text{and} \quad 2a = y+0,$$

so we get that $x=2a-10$ and $y=2a$. Then, by the distance formula, we have

$$(2a-10)^2+(2a)^2=20^2 \implies 2a^2-10a-75=0.$$

Finally, by Vieta's Formulas, we get the requested answer of $\frac{10}{2} = \boxed{\text{(A) } 5}$. ■

Problem 17:

(pog) For positive integers n, let the nth triangular number be the sum of the first n positive integers. For how many integers n between 1 and 100, inclusive, does the nth triangular number have the same last digit as the product of the first n triangular numbers?

(A) 11 **(B)** 12 **(C)** 20 **(D)** 21 **(E)** 22

Answer (E):

Note that the nth triangular number is equal to $\frac{n(n+1)}{2}$. As well, note that for all $n \geq 4$, the first n triangular numbers will be divisible by the fourth triangular number, 10, so they will have a last digit of 0.

Thus, for $n \geq 4$, we want $\frac{n(n+1)}{2}$ to have a last digit of 0; (e.g., $\frac{n(n+1)}{2}$ is divisible by 10).

Consequently, we can set $n(n+1) = 2k$, where k is a multiple of 10. This is equivalent to $n = 20\left(\frac{1}{10}k\right)$, and since k is a multiple of 10, we have that $\frac{1}{10}k$ is an integer.

Hence, $n(n+1)$ must be divisible by 20. Since exactly one of n and $n+1$ will be even, one of them must contain all of the powers of 2 in 20, and the other one cannot contain any powers of 2.

Case 1: n is divisible by both 4 and 5

This happens when n has a remainder of 0 when divided by 20.

Case 2: $n+1$ is divisible by both 4 and 5.

This happens when n has a remainder of 19 when divided by 20.

Case 3: n is divisible by 4 and $n+1$ is divisible by 5.

This happens when n has a remainder of 4 when divided by 20.

Case 4: n is divisible by 5 and $n+1$ is divisible by 4.

This happens when n has a remainder of 5 when divided by 20.

Thus, if $n \geq 4$, we get that n must have a remainder of 0, 4, 15, or 19 when divided by 20.

Note that, for each of the intervals $[21, 40]$, $[41, 60]$, $[61, 80]$, $[81, 100]$, there are 4 possible values of n. If $4 \le n \le 20$, then the possible values of n are 4, 15, 19, and 20. Finally, testing $n < 4$, we also see that $n = 1$ and $n = 2$ satisfy our condition. Hence, the requested answer is $4 \cdot 5 + 2 = \boxed{\textbf{(E) } 22}$. ■

Problem 18:

(HrishiP) Let x and y be real numbers such that

$$|x - |y - x|| = 1,$$
$$|y - |x - y|| = 2.$$

What is the largest possible value of $x + y$?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Answer (C):

First, suppose that $y > x$. Then, the first equation is $|2x - y| = 1$ and the second equation is $|x| = 2$. Since it is optimal to have x positive, we let $x = 2$. Then, $|4 - y| = 1$, so either $y = 3$ or $y = 5$. We want the bigger solution, or $y = 5$.

Otherwise, suppose that $x > y$. Then, $|y - x| = x - y$ and $|x - y| = x - y$. Then, the first equation is $|y| = 1$ and the second equation is $|2y - x| = 2$. Letting $y = 1$, we have $|2 - x| = 2$. As $x = 4$ is optimal, we have $x + y = 5$.

Since $7 > 5$, we have that the requested answer is $x + y = \boxed{\textbf{(C) } 7}$. ■

Problem 19:

(DeToasty3) A car moves such that if there are n people in it, it moves at a constant rate of 4^n miles per hour. At noon, the car has 1 person in it and starts moving. After every mile, another person instantaneously gets in the car. How many people are in the car when the average speed the car has moved since noon reaches 17 miles per hour?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Answer (D):

We see that after the nth mile the car moves, the average speed it has moved, in miles per hour, is

$$\frac{n}{(0.25)^1 + (0.25)^2 + \cdots + (0.25)^n}.$$

We can see that as n increases, the denominator approaches

$$(0.25)^1 + (0.25)^2 + \cdots = \frac{0.25}{1-0.25} = \frac{1}{3}.$$

Testing out $n = 5$ gives

$$\frac{5}{(0.25)^1 + (0.25)^2 + \cdots + (0.25)^5} \gtrapprox 15,$$

and testing out $n = 6$ gives

$$\frac{6}{(0.25)^1 + (0.25)^2 + \cdots + (0.25)^6} \gtrapprox 18.$$

This means that the point in time where the average speed the car has moved reaches 17 miles per hour is after the 5th mile but before the end of the 6th mile, so there are $\boxed{\textbf{(D) } 6}$ people in the car at that time, as requested. ■

Problem 20:

(Awesome_guy) In rectangle $ABCD$, point A is reflected over diagonal $\overline{BD}$ to a point A'. If $A'B = A'C$ and $AA' = 6$, what is the area of rectangle $ABCD$?

(A) 18 **(B)** $8\sqrt{6}$ **(C)** $12\sqrt{3}$ **(D)** 21 **(E)** $9\sqrt{6}$

Answer (C):

Call the intersection of segment $\overline{AA'}$ and diagonal $\overline{BD}$ point P. We then have that $BP : BD = 1 : 4$. This is because P is the midpoint of segment $\overline{AA'}$ due to reflections, and the foot of the perpendicular from A' to side $\overline{BC}$ is the midpoint of side $\overline{BC}$ because $A'B = A'C$. Now, we let $BP = x$. Then, we have that $DP = 3x$. We have that $\overline{AP} \perp \overline{BD}$ due to reflections, so $AP = \frac{1}{2} \cdot AA' = 3$. By the Pythagorean Theorem,

$$\begin{aligned} AB &= \sqrt{BP^2 + AP^2} = \sqrt{x^2 + 9}, \\ AD &= \sqrt{DP^2 + AP^2} = \sqrt{9x^2 + 9}, \\ BD &= \sqrt{AB^2 + AD^2} = \sqrt{10x^2 + 18}. \end{aligned}$$

However, we also see that $BD = 4x$, so we have

$$\sqrt{10x^2 + 18} = 4x \implies 10x^2 + 18 = 16x^2 \implies x^2 = 3 \implies x = \sqrt{3}.$$

Finally, recall that $AB = \sqrt{x^2 + 9}$ and $AD = \sqrt{9x^2 + 9}$. Plugging in our obtained value for x gives us $AB = 2\sqrt{3}$ and $AD = 6$, so the area of rectangle $ABCD$ is

$$AB \cdot AD = 2\sqrt{3} \cdot 6 = \boxed{\textbf{(C) } 12\sqrt{3}},$$

as requested. ■

Problem 21:

(DeToasty3) Richard thinks of a positive integer n and writes the base ten representations of $n!$ and $(n+1)!$ on a board. He then erases the zeroes to the right of the last nonzero digit of each number (if any exist), resulting in two numbers a and b. If one of a and b is 4 times the other, what is the sum of all possible values of n less than 1000?

(A) 315 **(B)** 441 **(C)** 656 **(D)** 714 **(E)** 819

Answer (D):

From the given condition, we have that

$$\frac{(n+1)!}{n!} = n + 1 = 4 \cdot 10^k \text{ or } \frac{1}{4} \cdot 10^k,$$

where k is an integer such that either $4 \cdot 10^k$ or $\frac{1}{4} \cdot 10^k$ is an integer (because $n+1$ is an integer). We obtain

$$n + 1 \in \{4, 25, 40, 250, 400\} \implies n \in \{3, 24, 39, 249, 399\}.$$

It is easy to see that these all work, so the answer is

$$3 + 24 + 39 + 249 + 399 = \boxed{\textbf{(D) } 714},$$

as requested. ■

Problem 22:

(DeToasty3) In the xy-plane are perpendicular lines $y = ax + d$ and $y = bx + c$, where a, b, c, and d are real numbers in a geometric progression in that order. If the two lines and the line $y = \frac{3}{2}x$ pass through a common point, what is the least possible value of $a + b + c + d$?

(A) $\frac{3}{2}$ **(B)** $\frac{51}{32}$ **(C)** $\frac{13}{8}$ **(D)** $\frac{111}{64}$ **(E)** $\frac{7}{4}$

Answer (B):

Note that by the condition that $y = ax + d$ and $y = bx + c$ are perpendicular lines, we must have that $ab = -1 \implies b = -\frac{1}{a}$. Since we are also given that a, b, c, and d form a geometric progression in that order, we have that $c = \frac{1}{a^3}$ and $d = -\frac{1}{a^5}$.

We now have the equations

$$y = ax - \frac{1}{a^5},$$
$$y = -\frac{1}{a}x + \frac{1}{a^3}.$$

Subtracting the second equation from the first equation, we get

$$0 = \left(a + \frac{1}{a}\right)x - \frac{1}{a^3} - \frac{1}{a^5} \implies \left(a + \frac{1}{a}\right)x = \frac{1}{a^3} + \frac{1}{a^5} \implies x = \frac{1}{a^4},$$

where we can do the last step because 0 is an unattainable value for $a + \frac{1}{a}$ when a is real. Substituting this value of x into either equation, we get that $y = \frac{a^2-1}{a^5}$.

Next, from the fact that this point must be on the line $y = \frac{3}{2}x$, we get the equation

$$\frac{a^2 - 1}{a^5} = \frac{3}{2a^4} \implies a^2 - 1 = \frac{3}{2}a \implies a^2 - \frac{3}{2}a - 1 = 0.$$

Now, by factoring the quadratic equation into

$$\left(a + \frac{1}{2}\right)(a - 2) = 0,$$

we get that $a \in \{-\frac{1}{2}, 2\}$.

Finally, we have that

$$a + b + c + d = a - \frac{1}{a} + \frac{1}{a^3} - \frac{1}{a^5},$$

so intuitively, we see that $a = 2$ will give us the smallest possible value of $a+b+c+d$. Plugging in $a = 2$, we get that the smallest possible value of $a + b + c + d$ is

$$2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{32} = \boxed{\text{(B) } \frac{51}{32}},$$

as requested. ■

Problem 23:

(DeToasty3) There are 15 people in a room, where everyone shakes hands with a positive number of other people in the room exactly once. If exactly 6 people shook 1 hand, exactly 5 people shook between 2 and 4 hands, inclusive, exactly 1 person shook 8 hands, and exactly 1 person shook 14 hands, what is the least possible total number of handshakes?

(A) 24 **(B)** 25 **(C)** 26 **(D)** 27 **(E)** 28

Answer (C):

We see that 1 person shook 14 hands, which means that we can remove that person. Also, we see that 6 people shook 1 hand, which means that they must have shaken hands with the person who shook 14 hands, so we can remove these 6 people.

Now, we are left with the 5 people who shook between 2 and 4 hands, inclusive, the 1 person who shook 8 hands, and 2 more people who are not specified. Next, the person who shook 8 hands must have shaken hands with the person who shook 14 hands as well as the other 7 remaining people besides themselves. Then, we can remove the person who shook 8 hands.

Now, we are left with the 5 who shook between 2 and 4 hands, inclusive, and 2 more people who are not specified. Note that these $5+2=7$ remaining people must have shaken hands with the person who shook 14 hands and the person who shook 8 hands.

Finally, to minimize the total number of handshakes, we must have that the 2 unspecified people shook 5 hands each, including a handshake with each other, and 2 handshakes each with two of the 5 people who shook between 2 and 4 hands, inclusive. Thus, the requested least possible total number of handshakes is $14+7+1+4=\boxed{\textbf{(C) } 26}$. ■

Problem 24:

(HrishiP) In right $\triangle ABC$ with $BC=7$ and a right angle at A, let the midpoint of side $\overline{AB}$ be D. Suppose that there exists a point E on the circumference of the circumcircle of $\triangle ABC$ such that $\triangle CDE$ is equilateral. What is the side length of $\triangle CDE$?

(A) $\sqrt{21}$ **(B)** $2\sqrt{7}$ **(C)** $\frac{7\sqrt{21}}{6}$ **(D)** $\frac{7\sqrt{10}}{4}$ **(E)** $3\sqrt{6}$

Answer (B):

Let O denote the midpoint of $\overline{BC}$, which is also the center of the circumcircle of $\triangle ABC$. We see that $CD=DE$, $DO=DO$, and $CO=EO$, so we have that $\triangle DOC \cong \triangle DOE$. Note that this means that $\triangle DOC$ and $\triangle DOE$ are reflections across line DO, and so C and E are reflections across line DO. Note that D is the midpoint of $\overline{AB}$, and O is the midpoint of $\overline{BC}$, so $\overline{DO} \parallel \overline{AC}$. Since $\angle BAC=90°$, we have that $\angle BDO=90°$.

Therefore, we find that A and B are reflections across line DO. Now, since C and E are reflections across line DO, and A and B are reflections across line DO, we have that $\angle ABE = \angle BAC = 90°$. Now, we can deduce that $\angle BEC = 180° - \angle BAC = 90°$, and similarly, $\angle ACE = 90°$.

Let line DO intersect segment $\overline{CE}$ at a point F. If we let $AC = 2x$ and $AB = 2y$, we now see that $BD = EF = y$, $DO = x$, $FO = x$, and $CE = 2y$. Using the Pythagorean Theorem on triangle $\triangle DEF$, we get

$$DE^2 = CE^2 = (2y)^2 = DF^2 + EF^2 = (2x)^2 + y^2 \implies 3y^2 = 4x^2.$$

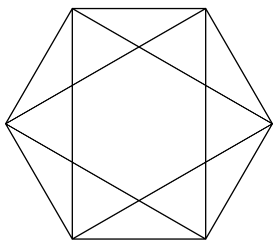
Furthermore, by the Pythagorean Theorem on triangle $\triangle ABC$, we get

$$BC^2 = 7^2 = (2x)^2 + (2y)^2 = 4x^2 + 4y^2 = 3y^2 + 4y^2 \implies y^2 = 7 \implies y = \sqrt{7}.$$

Thus, the requested side length of $\triangle CDE$ is $CE = 2y = \boxed{\textbf{(B) } 2\sqrt{7}}$. ■

Problem 25:

(DeToasty3) Each of the six vertices of the regular hexagon shown below is labeled with either a 1 or a 2. Some diagonals of the hexagon are drawn, and each of the six points of intersection is labeled with either a 2, a 3, or a 4. In how many ways can the 12 points be labeled such that for every drawn diagonal of the hexagon, the sum of the numbers on its two endpoints is *not* equal to either of the numbers on the two points of intersection of the diagonal? Rotations and reflections are considered distinct.



(A) 502 **(B)** 514 **(C)** 526 **(D)** 538 **(E)** 550

Answer (C):

Let us label the vertices $ABCDEF$, going clockwise. Note that any two adjacent diagonals, e.g. AC and BD, are home to exactly one of the intersection points. If the sums are the same, then there are two choices for the number. If they are different, then there is one choice for the number. Now, let us perform casework on triangles ACE and BDF.

For each of the triangles ACE and BDF, there are a total of four possible ways for the sides to have certain sums. If we have $1, 1, 1$, our sums are $2, 2, 2$. If we have $1, 2, 2$, our sums are $3, 3, 4$. If we have $1, 1, 2$, our sums are $2, 3, 3$. If we have $2, 2, 2$, our sums are $4, 4, 4$.

Case 1: ACE and BDF both have sum $2, 2, 2$ or both have sum $4, 4, 4$.

In this case, we either have $2, 2, 2, 2, 2, 2$ or $4, 4, 4, 4, 4, 4$. Then, the intersection points each have two choices, for a total of $2^6 = 64$ ways for each case, $2 \cdot 64 = 128$ cases in total.

Case 2: ACE and BDF have opposite sums $2, 2, 2$ and $4, 4, 4$.

In this case, we have $2, 4, 2, 4, 2, 4$ in some order. Then, the intersection points each have one choice, for a total of $1^6 = 1$ ways for this case. Multiplying by 2 to account for rotations being distinct, we get $2 \cdot 1 = 2$.

Case 3: ACE is X, X, X and BDF is $Y, 3, 3$, where X and Y are distinct and are either 2 or 4.

We will multiply by 2 at the end to account for switching X and Y. In this case, we have $XYX3X3$ in some order, which can be rotated in 6 distinct ways, so every adjacent sum is different, so our total here is $1^6 = 1$, and multiplying by 2 gives us $1 \cdot 2 = 2$. Finally, multiplying by 6 gives us $6 \cdot 2 = 12$.

Case 4: ACE is X, X, X and BDF is $X, 3, 3$, where X is either 2 or 4.

We will multiply by 2 at the end to account for X having two possibilities. In this case, we have $XXX3X3$ in some order, which can be rotated in 6 different ways. We have two adjacent pairs with the same sum and four adjacent pairs with different sums, so our total here is $2^2 \cdot 1^4 = 4$, and multiplying by 2 gives us $2 \cdot 4 = 8$. Finally, multiplying by 6 gives us $6 \cdot 8 = 48$.

Case 5: ACE is $X, 3, 3$ and BDF is $Y, 3, 3$, where X and Y are distinct and are either 2 or 4.

We have two subcases here: the Y is adjacent to the X, or the Y is opposite from the X.

Subcase 1: The Y is opposite to the X

When the Y is opposite from the X, we have $X, 3, 3, Y, 3, 3$ in some order. We have two adjacent pairs with the same sum and four adjacent pairs with different sums, so our total here is $2^2 \cdot 1^4 = 4$. We also have that reflections are distinct, so we multiply by 6 here to get $6 \cdot 4 = 24$.

Subcase 2: The Y is adjacent from the X

When the Y is adjacent to the X, we either have $X, Y, 3, 3, 3, 3$ in some order or $Y, X, 3, 3, 3, 3$ in some order, where each has 6 different rotations. Here, we have three adjacent pairs with the same sum and three adjacent pairs with different sums, so our total here is $2^3 \cdot 1^3 = 8$, and multiplying by 2, we get $2 \cdot 8 = 16$. Finally, multiplying by 6 gives us $6 \cdot 16 = 96$.

Thus, our total for Case 5 is $24 + 96 = 120$.

Case 6: ACE is $X, 3, 3$ and BDF is $X, 3, 3$, where X is either 2 or 4.

We will multiply by 2 at the end to account for X having two possibilities. We have two subcases here: the Xs are opposite, or the Xs are adjacent.

Subcase 1: The Xs are opposite

When the Xs are opposite, we have $X, 3, 3, X, 3, 3$ in some order, where there are 3 different rotations. We have two adjacent pairs with the same sum and four adjacent pairs with different sums, so our total here is $2^2 \cdot 1^4 = 4$. Finally, multiplying by 3 gives us $3 \cdot 4 = 12$.

Subcase 2: The Xs are adjacent

When the Xs are adjacent, we have $X, X, 3, 3, 3, 3$ in some order, where there are 6 different rotations. We have four adjacent pairs with the same sum and two adjacent pairs with different sums, so our total here is $2^4 \cdot 1^2 = 16$. Finally, multiplying by 6 gives us $6 \cdot 16 = 96$.

Thus, our total for Case 6 is $12 + 96 = 108$. But we have to multiply by 2 since X can either be 2 or 4. Therefore, our total is $2 \cdot 108 = 216$.

Thus, the requested answer is $128 + 2 + 12 + 48 + 120 + 216 = \boxed{\textbf{(C) } 526}$. ■