# Official Solutions

De Mathematics Competitions

3rd Annual

# DMC 10 A

Friday, May 27, 2022

This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.



Questions and complaints about this competition should be sent by private message to

**DeToasty3**.

# Answer Key:

| | | | | |
|---|---|---|---|---|
| 1. (C) | 2. (D) | 3. (B) | 4. (B) | 5. (B) |
| 6. (E) | 7. (C) | 8. (C) | 9. (A) | 10. (D) |
| 11. (B) | 12. (E) | 13. (D) | 14. (C) | 15. (A) |
| 16. (B) | 17. (D) | 18. (D) | 19. (D) | 20. (A) |
| 21. (E) | 22. (D) | 23. (C) | 24. (D) | 25. (A) |

# Problem 1:

**(pog)** A red container is filled with water. If 20% of the water in the red container is poured into an empty blue container, what is the ratio of the amount of water in the blue container to the amount of water in the red container?

**(A)** $1:6$ **(B)** $1:5$ **(C)** $1:4$ **(D)** $1:3$ **(E)** $1:2$

**Answer (C):**

We get that $100\% - 20\% = 80\%$ of the water is in the red container and $20\%$ of the water is in the blue container, so the requested ratio is $\boxed{\textbf{(C)}\ 1:4}$. ■

# Problem 2:

**(DeToasty3)** What is the smallest positive integer $n$ such that

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{n}$$

is less than 1?

**(A)** 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

**Answer (D):**

Computing, we get that $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$. Thus, we have that $\frac{5}{6}+\frac{1}{n}<1$, so $\frac{1}{n}<\frac{1}{6}$. Consequently, $n>6$, so the smallest positive integer value of $n$ is $\boxed{\textbf{(D)}\ 7}$. ■

# Problem 3:

**(pog)** Let $a$ and $b$ be real numbers. If the average of the numbers 24, 29, $a$, and $b$ is 6, what is the average of the numbers 24, 29, $a+1$, and $b+1$?

**(A)** $6\frac{1}{4}$ **(B)** $6\frac{1}{2}$ **(C)** $6\frac{3}{4}$ **(D)** $7\frac{1}{4}$ **(E)** $7\frac{1}{2}$

**Answer (B):**

We have that $\frac{24+29+a+b}{4} = 6$, so $24 + 29 + a + b = 24$. Hence, $24 + 29 + (a + 1) + (b + 1) = 24 + 1 + 1 = 26$, so the requested average is equal to $\frac{26}{4} = \boxed{\textbf{(B) } 6\frac{1}{2}}$. ■

## Problem 4:

**(pog)** Let $n$ be the smallest positive integer whose digits sum to 2022. What is the sum of the digits of $n + 1$?

**(A)** 6 **(B)** 7 **(C)** 223 **(D)** 2014 **(E)** 2023

**Answer (B):**

We want $n$ to have as many digits that are equal to 9 as possible. To minimize $n$ given that the sum of its digits is 2022, it should have as few digits as possible. Thus, the digits should be as big as possible with larger digits to the right of smaller digits. Since $2022 = 6 + 9 \cdot 224$, the smallest such $n$ is 6 followed by 224 9s. Then, $n + 1$ is 7 followed by 224 0s, implying that the sum of the digits of $n + 1$ is $\boxed{\textbf{(B) } 7}$. ■

## Problem 5:

**(PhunsukhWangdu)** Joel's house and his office are located at $(0, 0)$ and $(4, 6)$ on the coordinate plane, respectively. Joel normally moves at 0.1 units per minute, but when he moves along the lines $x = 2$ and $y = 1$, he moves at 1 unit per minute. If Joel can only move one unit up or to the right at a time, what is the fewest number of minutes in which Joel can move from his house to his office?

**(A)** 28 **(B)** 37 **(C)** 46 **(D)** 64 **(E)** 100

**Answer (B):**

It is optimal for Joel to travel along $x = 2$ and $y = 1$ for as long as possible. If he moves to

$$(0, 1) \to (1, 1) \to (2, 1) \to (2, 2) \to \ldots \to (2, 6) \to (3, 6) \to (4, 6),$$

it will take 37 minutes. If Joel does not move along $x = 2$ from $(2, 1)$ to $(2, 6)$, he will have to move at least 4 times (not along the lines $x = 2$ or $y = 1$), which will take at least 40 minutes. Consequently, our answer is

**(B) 37**. ■

## Problem 6:

**(pog & DeToasty3)** There are 10 students in a classroom with 12 chairs. Before lunch, each student sat on a different chair. After lunch, each student randomly chose a chair to sit on. If no two students chose the same chair after lunch, what is the probability that every chair had been sat on at least once?

**(A)** $\frac{7}{24}$ **(B)** $\frac{11}{36}$ **(C)** $\frac{25}{66}$ **(D)** $\frac{14}{33}$ **(E)** $\frac{15}{22}$

**Answer (E):**

Suppose the chairs are numbered $1, 2, \ldots, 12$. Without loss of generality, assume that the students sit in the chairs $1, 2, \ldots, 10$ before lunch. Then, for every chair to have been sat on at least once after lunch, 2 students must sit in chairs 11 and 12, while the remaining 8 each sit in one of the first 10 seats. There are $\binom{10}{8} = 45$ ways to determine such a seating. Since there are $\binom{12}{10} = 66$ possible seatings, the desired probability is $\frac{45}{66} = \boxed{\textbf{(E)}\ \frac{15}{22}}$. ■

## Problem 7:

**(DeToasty3)** Bo writes down all the divisors of 144 on a board. He then erases some of the divisors. If no two of the divisors left on the board have a product divisible by 18, what is the least number of divisors Bo could have erased?

**(A)** 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

**Answer (C):**

Clearly, Bo should erase all divisors on the board that are divisible by 18. This leaves $1, 2, 4, 8, 16, 3, 6, 12, 24, 48$, and 9 left on the board, accounting for 4 divisors erased so far.

Consider the set $S = \{3, 6, 12, 24, 48\}$. For any two distinct integers from $S$, at least one integer will be even. In addition, any integer in $S$ is divisible by 3. Hence, the product of any two distinct integers in $S$ is divisible by 18. Thus, at most 1 element of $S$ can be left on the board, implying at least 4 more divisors will have to be erased.

If 9 is not erased, then $2, 4, 8$, and 16 must be erased. Thus, at least 1 more divisor will have to be erased.

Consequently, at least $4 + 4 + 1 = 9$ divisors must be erased. Indeed, if the divisors left on the board are 1, 2, 4, 8, 16, and 3, no two of these divisors have a product divisible by 18 because there is only one integer on the board divisible by 3 and that integer is not divisible by 9. Hence, the requested minimum is $\boxed{\textbf{(C)}\ 9}$. ■

## Problem 8:

**(pog)** In the expression

$$(\_\_\_\_ + \_\_\_\_ + \_\_\_\_) - (\_\_\_\_ + \_\_\_\_ + \_\_\_\_)$$

each blank is to be filled in by one of the digits 1, 2, 3, 4, 5, or 6, with each digit being used once. How many different values can be obtained?

**(A)** 5 **(B)** 8 **(C)** 10 **(D)** 14 **(E)** 19

**Answer (C):**

Let $x$ be the integer inside the first set of parentheses and $y$ be the integer inside the second set of parentheses. Observe that

$$x + y = 1 + 2 + 3 + 4 + 5 + 6 = 21.$$

In addition, since $x$ and $y$ are integers, $x + y$ and $x - y$ have the same parity, implying that $x - y$ is odd. This implies that $x - y$ can never equal 0. Also, by switching the numbers in the first 3 blanks with the numbers in the last 3 blanks, the value $-(x - y)$ can be obtained. Thus, it suffices to find the number of possible positive values of $x - y$ and then multiply by 2 to account for negative values.

Observe that $x \leq 4 + 5 + 6 = 15$ and $y \geq 1 + 2 + 3 = 6$. Hence, $x - y \leq 9$. Since $x - y$ is odd, the only possible positive values of $x - y$ are 1, 3, 5, 7, and 9. Indeed, each of these values can be obtained:

$$(1+4+6)-(2+3+5) = 1, \quad (1+5+6)-(2+3+4) = 3, \quad (2+5+6)-(1+3+4) = 5,$$

$$(3 + 5 + 6) - (1 + 2 + 4) = 7, \quad (4 + 5 + 6) - (1 + 2 + 3) = 9.$$

To account for negative values, there are $2 \cdot 5 = \boxed{\textbf{(C)}\ 10}$ different obtainable values. ■

## Problem 9:

**(DeToasty3 & pog)** Let $a$, $b$, and $c$ be positive real numbers such that the ratio of $a$ to $bc$ is $1 : 3$, the ratio of $b$ to $ac$ is $1 : 12$, and the ratio of $c$ to $a + b$ is $1 : 8$. What is $b + c$?

**(A)** 22 **(B)** 24 **(C)** 26 **(D)** 28 **(E)** 30

**Answer (A):**

We get that $bc = 3a$ and $ac = 12b$. Hence,

$$bc \cdot ac = abc^2 = 3a \cdot 12b = 36ab,$$

so thus $c^2 = 36$. Since $c$ is positive, we get that $c = 6$. Thus, substituting $c = 6$ into our three ratios, we have that $6b = 3a$, $6a = 12b$, and $a+b = 48$. Solving, we get that $a = 32$ and $b = 16$. Consequently, $b + c = 16 + 6 = \boxed{\textbf{(A) } 22}$. ■

## Problem 10:

**(DeToasty3)** In how many ways can a non-empty subset $A$ of $\{1, 2, 3, 4\}$ and a non-empty subset $B$ of $\{3, 4, 5, 6\}$ be chosen so that $A$ is a subset of $B$?

**(A)** 6 **(B)** 12 **(C)** 16 **(D)** 20 **(E)** 36

**Answer (D):**

Clearly, $A$ cannot be a subset of $B$ if $A$ contains 1 or 2. Hence, $A$ is a non-empty subset of $\{3, 4\}$.

If $A = \{3\}$, then $B$ must contain $\{3\}$ merged with some (possibly empty) subset of $\{4, 5, 6\}$. There are $2^3$ such subsets.

Similarly, if $A = \{4\}$, then $B$ must contain $\{4\}$ merged with some (possibly empty) subset of $\{3, 5, 6\}$. There are $2^3$ such subsets.

Lastly, if $A = \{3, 4\}$, then $B$ must contain $\{3, 4\}$ merged with some (possibly empty) subset of $\{5, 6\}$. There are $2^2$ such subsets.

In total, there are $2^3 + 2^3 + 2^2 = \boxed{\textbf{(D) } 20}$ ways. ■

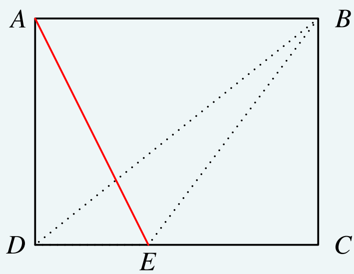
## Problem 11:

**(DeToasty3)** In rectangle $ABCD$ with $AB = 5$ and $BC = 4$, let $E$ be a point on side $\overline{CD}$. Given that segment $\overline{AE}$ bisects $\angle BED$, what is the length of $\overline{AE}$?

**(A)** $3\sqrt{2}$ **(B)** $2\sqrt{5}$ **(C)** $3\sqrt{3}$ **(D)** $2\sqrt{7}$ **(E)** $\sqrt{30}$

**Answer (B):**

Consider the following diagram:




We know that $\overline{AB} \parallel \overline{CD}$ and that $\angle BEA = \angle DEA$, so $\angle BEA = \angle BAE$. Consequently, $AB = BE = 5$. Since $BE = 5$, $BC = 4$, and $\angle BCE = 90^\circ$, we have that $EC = 3$, which implies that $DE = 5 - EC = 2$. Therefore,

$$AE = \sqrt{AD^2 + DE^2} = \sqrt{16 + 4} = \sqrt{20} = \boxed{\textbf{(B) } 2\sqrt{5}}.$$

■

## Problem 12:

**(pog)** The product of the perfect square divisors of $12^{12}$ is equal to $12^n$, where $n$ is a positive integer. What is the sum of the digits of $n$?

**(A)** 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

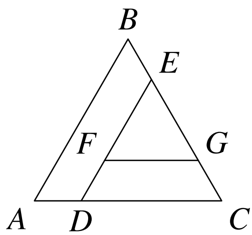
**Answer (E):**

We get that $12^{12} = 2^{24} \cdot 3^{12}$. Since $12^{12}$ itself is a perfect square, we may pair the perfect square divisors so that they multiply to $12^{12}$, except for $12^6$.

Since $12^{12}$ has $13 \cdot 7 = 91$ perfect square divisors, the requested product is $(12^{12})^{45} \cdot 12^6 = 12^{546}$ and $n = 546$, so our answer is $5 + 4 + 6 = \boxed{\textbf{(E) } 15}$. ■

## Problem 13:

**(pog & DeToasty3)** In equilateral $\triangle ABC$, let $D$ and $E$ be on $\overline{AC}$ and $\overline{BC}$, respectively, such that $CD = CE$, and let $F$ and $G$ be on $\overline{DE}$ and $\overline{CE}$, respectively, such that $EF = EG$. If the perimeters of $CDFG$ and $ABED$ are 17 and 22, respectively, and $AD = DF$, what is the perimeter of $\triangle EFG$?




**(A)** 12 **(B)** $12\frac{3}{4}$ **(C)** $13\frac{1}{2}$ **(D)** $14\frac{1}{4}$ **(E)** 15

**Answer (D):**

Note that since $CD = CE$ and $EF = EG$, we have that $\triangle EFG$ and $\triangle CDE$ are equilateral by similar triangles. Then, let $EF = s$ and $AD = DF = x$. We have that the perimeter of $CDFG$ is equal to

$$CD + DF + FG + GC = (s + x) + x + s + x = 2s + 3x = 17,$$

and the perimeter of $ABED$ is equal to

$$AB + BE + ED + DA = (s + 2x) + x + (s + x) + x = 2s + 5x = 22.$$

Subtracting the two equations gives us $2x = 5 \implies x = \frac{5}{2}$. Then, it follows that $s = \frac{19}{4}$. Finally, the perimeter of $\triangle EFG$ is equal to $3s$, or $3 \cdot \frac{19}{4} = \boxed{\textbf{(D)}\ 14\frac{1}{4}}$. ■

## Problem 14:

**(DeToasty3)** Ryan has 5 pieces of taffy and 6 pieces of gum, which he randomly distributes to three boys all at once. If each boy ends up with at least one piece of each sweet (taffy and gum), what is the probability that a boy ends up with more pieces of taffy but fewer pieces of gum than each of the other boys?

**(A)** $\frac{1}{20}$ **(B)** $\frac{1}{12}$ **(C)** $\frac{1}{10}$ **(D)** $\frac{1}{6}$ **(E)** $\frac{1}{5}$

**Answer (C):**

By stars and bars, there are $\binom{4}{2} \cdot \binom{5}{2} = 60$ ways to distribute the sweets.

If one of the boys gets more taffy but fewer gum, then the taffy and gum must be distributed like so:

$$(t, g) = (\{3, 1, 1\},\ \ \{1, 2, 3\}) \quad \text{or} \quad (\{3, 1, 1\},\ \ \{1, 3, 2\}).$$

For each of the 2 arrangements, there are 3 ways to choose which boy gets more taffy but fewer gum, and then the distribution of sweets will be uniquely determined from there. Hence, the requested probability is $\frac{3 \cdot 2}{60} = \boxed{\textbf{(C)}\ \frac{1}{10}}$.

■

## Problem 15:

**(pog)** Let $a$, $b$, and $c$ be real numbers which satisfy

$$\begin{aligned} a + b + c &= 1, \\ a + |b| + |c| &= 4, \\ |a| + b + |c| &= 5, \\ |a| + |b| + c &= 8. \end{aligned}$$

What is $a^2 + b^2 + c^2$?

**(A)** $\frac{53}{2}$ **(B)** $\frac{55}{2}$ **(C)** $\frac{57}{2}$ **(D)** $\frac{59}{2}$ **(E)** $\frac{61}{2}$

**Answer (A):**

Taking advantage of symmetry, we add the equations together, giving

$$2(a + b + c + |a| + |b| + |c|) = 18.$$

Thus, $|a| + |b| + |c| = 8$. We get that $|a| - a = 4$ and $|b| - b = 3$, so thus $(a, b) = \left(-2, -\frac{3}{2}\right)$. Since $a + b + c = 1$, we get that $c = \frac{9}{2}$, so our answer is

$$(-2)^2 + \left(-\frac{3}{2}\right)^2 + \left(\frac{9}{2}\right)^2 = 4 + \frac{9}{4} + \frac{81}{4} = \frac{106}{4} = \boxed{\textbf{(A)}\ \frac{53}{2}}.$$

■

## Problem 16:

**(HrishiP & pog)** Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Each minute, Ryan randomly chooses one of the numbers still in $S$ and removes that number and all numbers *not* relatively prime to that number from $S$. Ryan continues until $S$ is empty, at which point he stops. The expected number of minutes in which he stops is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. What is $m + n$?

**(A)** 11 **(B)** 35 **(C)** 41 **(D)** 49 **(E)** 67

**Answer (B):**

Observe that 1, 5, and 7 are relatively prime to all other elements of $S$. Thus, the only way 1, 5, and 7 can be removed is if Ryan chooses them, always contributing 3 minutes.

It then suffices to determine the expected number of minutes for Ryan to remove all elements of $S' = \{2, 3, 4, 6, 8, 9\}$. Consider cases on the first element Ryan removes from $S'$.

If Ryan removes 2, 4, or 8 first, which occurs with probability $\frac{1}{2}$, $S'$ will only contain 3 and 9. Removing one of these elements will cause the other to be removed, so it will take 2 minutes for Ryan to remove the elements of $S'$.

If Ryan removes 3 or 9 first, which occurs probability $\frac{1}{3}$, $S'$ will only contain 2, 4, and 8. Removing one of these elements will cause the other two to be removed, so it will take 2 minutes for Ryan to remove the elements of $S'$.

If Ryan removes 6 first, which occurs probability $\frac{1}{6}$, all elements of $S'$ will be removed. Thus, it will take 1 minute for Ryan to remove the elements of $S'$.

Hence, the expected number of minutes for Ryan to remove all elements of $S'$ is $2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} + 1 \cdot \frac{1}{6} = \frac{11}{6}$. Adding on the 3 minutes required for Ryan to remove 1, 5, and 7, the expected number of minutes is $3 + \frac{11}{6} = \frac{29}{6}$, for an answer of $29 + 6 = \boxed{\textbf{(B) } 35}$. ■

## Problem 17:

**(DeToasty3)** In trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$, $AB = 3$, $CD = 7$, and $AD = BC$, let $M$ be the midpoint of side $\overline{BC}$. If the circle with diameter $\overline{DM}$ is tangent to line $AB$, what is the length of the altitude from $\overline{AB}$ to $\overline{CD}$?

**(A)** $2\sqrt{3}$ **(B)** $\sqrt{15}$ **(C)** 4 **(D)** $3\sqrt{2}$ **(E)** $2\sqrt{5}$

**Answer (D):**

Let $O$ denote the center of the circle. Define $E$, $F$, and $G$ as the feet of the altitudes to line $AB$ from $D$, $O$, and $C$, respectively. Clearly, $CDEG$ is a rectangle, implying that $EG = 7$. In addition, by symmetry, $AE = BG$, implying that $AE = BG = 2$ from $AB = 3$. The length of the altitude from $\overline{AB}$ to $\overline{CD}$ is given by the length of $DE$.

Clearly, $\angle BMH = \angle CMD$ and by parallel lines, $\angle BHM = \angle CDM$. Thus, $\triangle BHM \sim \triangle CDM$. Furthermore, since $BM = MC$, $\triangle BHM$ and $\triangle CDM$ are congruent. Hence, $BH = DC = 7$. Then, $EH = EA + AB + BH = 12$.

Observe that

$$FO : OH = FO : (OM + MH) = FO : (OM + DM) = 1 : 3,$$

since $O$ is the midpoint of $DM$ and $OF = OD = OM$. Then, by Pythagorean Theorem, $FO : FH = 1 : 2\sqrt{2}$. Since $ED$ and $FO$ are parallel, $\triangle EDH \sim \triangle FOH$, implying that

$$\frac{ED}{EH} = \frac{FO}{FH} = \frac{1}{2\sqrt{2}} \implies ED = \boxed{\textbf{(D) } 3\sqrt{2}.}$$

■

## Problem 18:

**(john0512)** Alice has 6 coins, where one of them is special and always lands on heads, and the others each have a $\frac{1}{2}$ probability of landing on heads. Alice flips all 6 coins, and afterwards, she uniformly at random picks a coin that landed on heads. What is the probability that Alice picks the special coin?

**(A)** $\frac{7}{32}$ **(B)** $\frac{2}{7}$ **(C)** $\frac{8}{27}$ **(D)** $\frac{21}{64}$ **(E)** $\frac{1}{3}$

**Answer (D):**

For all integers $0 \le k \le 5$, the probability that exactly $k$ non-special coins that land on heads is given by $\binom{5}{k} \cdot \frac{1}{2^5}$. Afterwards, there are $k+1$ coins that landed on heads, so the probability that Alice picks the special coin is $\frac{1}{k+1}$. Hence, the desired probability is $\binom{5}{k} \cdot \frac{1}{2^5} \cdot \frac{1}{k+1}$ summed from $0 \le k \le 5$:

$$\binom{5}{0} \cdot \frac{1}{2^5} \cdot \frac{1}{1} + \binom{5}{1} \cdot \frac{1}{2^5} \cdot \frac{1}{2} + \binom{5}{2} \cdot \frac{1}{2^5} \cdot \frac{1}{3} + \binom{5}{3} \cdot \frac{1}{2^5} \cdot \frac{1}{4} + \binom{5}{4} \cdot \frac{1}{2^5} \cdot \frac{1}{5} + \binom{5}{5} \cdot \frac{1}{2^5} \cdot \frac{1}{6}$$

$$= \frac{1}{32}\left(1 + \frac{5}{2} + \frac{10}{3} + \frac{10}{4} + \frac{5}{5} + \frac{1}{6}\right) = \frac{1}{32} \cdot \frac{21}{2} = \boxed{\textbf{(D)}\ \frac{21}{64}}.$$

## Problem 19:

**(DeToasty3)** A positive integer $n > 1$ is called *toasty* if for all integers $m$ with $1 \le m < n$, there exists a positive integer $k$ such that

$$\frac{m}{n} = \frac{k}{k+12}.$$

How many toasty integers are there?

**(A)** 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 6

**Answer (D):**

For a given pair of positive integers $(m, n)$,

$$\frac{m}{n} = \frac{k}{k+12} \implies m(k+12) = nk \implies k(n-m) = 12m \implies k = \frac{12m}{n-m}.$$

Hence, for $k$ to be an integer, $12m$ must be divisible by $n - m$.

For a positive integer $n \ge 2$ to be toasty, it is necessary for the divisibility to hold for $m = 1$. Hence, $n - 1$ divides 12, implying that the only $n$ that could be toasty are $2, 3, 4, 5, 7,$ and 13.

By the definition of a toasty integer, for $n = 2$ to be toasty, it sufficient for the divisibility to hold for $m = 1$. Since the divisibility holds, $n = 2$ is toasty.

Assume an integer $n \ge 3$ is toasty. Then, it is necessary for the divisibility to also hold for $m = 2$, implying that $n - 2$ divides 24. It then follows that 7 and 13 cannot be toasty.

Checking the other integers, $n = 3$ is toasty, since the divisibility holds for both $m = 1$ and $m = 2$. Similarly, $n = 4$ and $n = 5$ are toasty, since it can be verified that the divisibility holds for all integers $m$ such that $1 \le m < n$. Hence, all the toasty integers are 2, 3, 4, and 5, implying there are $\boxed{\textbf{(D)}\ 4}$ toasty integers.

## Problem 20:

**(DeToasty3)** Let $\lfloor r \rfloor$ denote the greatest integer less than or equal to a real number $r$. Let $N$ be the number of positive integers $n \le 100$ such that

$$\lfloor (n+1)\pi \rfloor - \lfloor n\pi \rfloor = 4.$$

What is the sum of the digits of $N$?

**(A)** 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

**Answer (A):**

For every positive integer $n$, $(n+1)\pi$ and $n\pi$ are real numbers that differ by $\pi$. Since $3 < \pi < 4$, it follows that $\lfloor (n+1)\pi \rfloor$ and $\lfloor n\pi \rfloor$ differ by either 3 or 4.

Let $a$ be the number of positive integers $n$ such that $1 \le n \le 100$ and $\lfloor (n+1)\pi \rfloor - \lfloor n\pi \rfloor = 3$. Define $b$ similarly for $\lfloor (n+1)\pi \rfloor - \lfloor n\pi \rfloor = 4$. Clearly, $a + b = 100$.

Observe that

$$\lfloor 101\pi \rfloor - \lfloor \pi \rfloor = (\lfloor 2\pi \rfloor - \lfloor \pi \rfloor) + (\lfloor 3\pi \rfloor - \lfloor 2\pi \rfloor) + \ldots + (\lfloor 101\pi \rfloor - \lfloor 100\pi \rfloor) = 3a + 4b.$$

Using $\pi \approx 3.1415$, we get that $\lfloor 101\pi \rfloor = 317$ and $\lfloor \pi \rfloor = 3$, so $3a + 4b = 314$. In combination with $a + b = 100$, this implies that $(a, b) = (86, 14)$. It then follows that $N = 14$, for which the requested sum is $\boxed{\textbf{(A) } 5}$. ■

## Problem 21:

**(DeToasty3 & HrishiP)** Each of the $N$ students in Mr. Ji's class took a 10-question quiz with questions $1, 2, \ldots, 10$. Suppose for every (possibly empty) subset of $\{1, 2, \ldots, 10\}$, there exists a student who got exactly those questions correct, and for every $i = 0, 1, 2, \ldots, 10$, if a student got $i$ questions correct, then of the students that got those same $i$ questions correct (including that student), the fraction of them that got over $i$ questions correct is $1 - 2^{i-10}$. If 3 students got a perfect score, what is the remainder when $N$ is divided by 100?

**(A)** 24 **(B)** 32 **(C)** 48 **(D)** 64 **(E)** 72

**Answer (E):**

I claim that the number of students who got exactly some (possibly empty) subset of questions $\{1, 2, \ldots, 10\}$ correct is 3. We will use induction, where we suppose that 3 students got a subset of $n$ problems correct, for some $1 \le n \le 10$. Our base case is $n = 10$. Then, since we are given that 3 students got a perfect score, we are done. Next, without loss of generality, suppose that 3 students got exactly questions $1, 2, \ldots, n$ correct, for $n = 10, 9, \ldots, k + 1$. We will show that 3 students got exactly questions $1, 2, \ldots, k$ correct. Note that for each non-empty subset of questions $\{k + 1, k + 2, \ldots, 10\}$, exactly 3 students got all of questions $1, 2, \ldots, k$ correct as well as the subset of questions $\{k + 1, k + 2, \ldots, 10\}$ correct. Since there are $2^{10-k} - 1$ subsets, there are $3 \cdot 2^{10-k} - 3$ students who got questions $1, 2, \ldots, k$ as well as other questions correct. Then, we have from the problem statement that if $S$ students got at least questions $1, 2, \ldots, k$ correct, then

$$\frac{3 \cdot 2^{10-k} - 3}{S} = 1 - 2^{k-10} \implies S = 3 \cdot 2^{10-k}.$$

Thus, we have that $S - (3 \cdot 2^{10-k} - 3) = 3$ students got exactly questions $1, 2, \ldots, k$ correct. Thus,

$$N = 3\left(\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \cdots + \binom{10}{10}\right) = 3 \cdot 1024 = 3072.$$

Thus, the answer is $\boxed{\textbf{(E) } 72}$. ■

## Problem 22:

**(HrishiP & DeToasty3)** In isosceles $\triangle ABC$ with $AB = AC = 4$ and $BC = 2$, let point $D$, distinct from $B$, be on side $\overline{AB}$ such that $CD = 2$. The circle passing through $B$, $C$, and $D$ intersects side $\overline{AC}$ and the line through $C$ perpendicular to $\overline{AB}$ at points $P$ and $Q$, respectively, both distinct from $C$. If $PQ^2$ is equal to $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers, what is $m + n$?

**(A)** 45 **(B)** 46 **(C)** 63 **(D)** 64 **(E)** 71

### Answer (D):

Let $\omega$ denote the circle passing through $B$, $C$, and $D$. In addition, let $M$ and $N$ be the midpoints of $BC$ and $BD$, respectively. By the symmetry of $\triangle ABC$ across $AM$, $DP$ is parallel to $BC$, implying that $BCPD$ is an isosceles trapezoid. In addition, by the symmetry of $\triangle BCD$ across $CN$, $CQ$ is a diameter of $\omega$. Hence, $\angle CBQ = \angle CPQ = 90^\circ$.

Clearly, $\angle BAM$ and $\angle BCN$ are complementary to $\angle ABC$. Hence, $\angle BAM = \angle QCB$. In addition, $\angle AMB = \angle CBQ = 90^\circ$, implying that $\triangle AMB \sim \triangle CBQ$. Observe that $BM = 1$. By Pythagorean Theorem on $\triangle ABM$, we have that $AM = \sqrt{15}$.

Using $\triangle AMB \sim \triangle CBQ$,

$$\frac{CQ}{CB} = \frac{AB}{AM} = \frac{4}{\sqrt{15}} \implies CQ = \frac{8}{\sqrt{15}}.$$

Observe that $\triangle ABC$ and $\triangle CBD$ are both isosceles and share $\angle ABC$. Since $\angle CDB = \angle CBD = \angle ABC = \angle ACB$, $\triangle ABC \sim \triangle CBD$. Hence,

$$\frac{BD}{BC} = \frac{BC}{AB} \implies BD = 1.$$

By Pythagorean Theorem on $\triangle PQC$, $PQ^2 = \frac{49}{15}$. The requested sum is $\boxed{\textbf{(D) } 64}$. ■

## Problem 23:

**(DeToasty3)** Let $ABCD$ be a rectangle with $AB > BC$. Let $E$ be a point on side $\overline{AD}$, and let $CEFG$ be the rectangle where $B$ is on side $\overline{FG}$. Let $H$ be the point on side $\overline{CD}$ such that $\overline{BH} \perp \overline{CE}$. If $CH = 2$, $DE = 3$, and the area of $CEFG$ is 48, then $AC = m\sqrt{n}$, where $m$ and $n$ are positive integers, and $n$ is not divisible by the square of any prime. What is $m + n$?

**(A)** 24 **(B)** 26 **(C)** 28 **(D)** 30 **(E)** 32

**Answer (C):**

Let $CE$ and $BH$ intersect at $I$. By right triangles $\triangle BCH$ and $\triangle CIH$, $\angle CBH$ and $\angle BHC$ are complementary and $\angle IHC$ and $\angle ICH$ are complementary. Hence, $\angle CBH = \angle DCE$. Since $\angle BCH = \angle CDE = 90^\circ$, $\triangle BCH \sim \triangle CDE$. Hence, $BC : CD = CH : DE = 2 : 3$.

Observe that $\triangle BCE$ shares the same base and height as rectangle $CEFG$. Hence, $[BCE] = \frac{1}{2}[CEFG]$. Similarly, $\triangle BCE$ shares the same base and height as rectangle $ABCD$, implying that $[BCE] = \frac{1}{2}[ABCD]$. Hence, $ABCD$ and $CEFG$ have the same area, implying that $[ABCD] = 48$.

Since $BC : CD = 2 : 3$, $BC = 4\sqrt{2}$ and $AB = CD = 6\sqrt{2}$. By Pythagorean Theorem, $AC = 2\sqrt{26}$. The requested sum is $\boxed{\textbf{(C) } 28}$. ■

## Problem 24:

**(DeToasty3)** How many ordered pairs of positive integers $(m, n)$ satisfy the following?

"There are exactly $m$ set(s) of 100 consecutive positive integers whose least element is less than 100 which contain exactly $m+1$ multiples of $n$."

**(A)** 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

**Answer (D):**

We have two cases for what $n$ is: $n < 100$ and $n \geq 100$. If $n < 100$, then consider the set of 100 consecutive positive integers whose smallest element is a multiple of $n$. Then, we have that there are $\lfloor \frac{99}{n} \rfloor + 1$ multiples of $n$ in this set because after the smallest element is a multiple of $n$, we have 99 more elements starting from $1 \pmod{n}$, so we will hit $\lfloor \frac{99}{n} \rfloor$ more multiples of $n$. There are also $\lfloor \frac{99}{n} \rfloor$ multiples of $n$ between 1 and 99, inclusive.

If $n$ is not a divisor of 99, then for each multiple of $n$ between 1 and 99, inclusive, we start with this multiple of $n$ as the smallest element of a set and then repeatedly subtract each element by 1 until we get to a point where either the largest element is a multiple of $n$ or where the smallest element is equal to 1. Thus, for each multiple of $n$ between 1 and 99, inclusive, we can always generate more sets (provided that $n > 1$).

If $n \geq 100$, then there can be at most 1 multiple of $n$ in a set. This is clearly not possible as $m$ would be either $-1$ or 0, neither of which are positive. Thus, we require that $n$ is a divisor of 99. Note that $99 = 3^2 \cdot 11$, which has $(2+1)(1+1) = 6$ divisors, so the answer is $\boxed{\textbf{(D) } 6}$. ■

## Problem 25:

**(HrishiP & DeToasty3)** In the $xy$-plane, a laser emanates from the origin with a path whose shape obeys $y = x^2$. Whenever the laser touches the line $y = x$, the path of the laser will reflect over the line parallel to the $x$-axis passing though where the laser last touched $y = x$, and whenever the laser touches the $x$-axis, the path of the laser will reflect over the $x$-axis. The graph below shows the path of the laser and its first 7 reflection points. If $N$ denotes the sum of the squares of the $x$-coordinates of the first 20 points where the laser intersects the $x$-axis (excluding the origin), what is the sum of the digits of $N$?

**(A)** 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

**Answer (E):**

Let $A_n$ denote the $n$th point where the laser touches $y = x$ after the origin. Similarly, let $B_n$ denote the $n$th point where the laser touches the $x$-axis after the origin.

When the laser is reflected across a line parallel to the $x$-axis, the resulting parabola formed by the laser's path will still have its vertex on the $y$-axis. The absolute value of the $x^2$ coefficient will not change after the reflection, but the sign of the $x^2$ coefficient will change. Hence, the laser's path from $A_n$ to $B_n$ or $B_n$ to $A_{n+1}$ for all positive integers $n$ can always be represented as the portion of the graph of $y = \pm x^2 + r$ for some real number $r$.

Clearly, $A_1 = (1, 1)$. The laser's path from $A_1$ to $B_1$ is a downward facing parabola that passes through $(1, 1)$, implying $y = -x^2 + 2$. This implies that $B_1 = (\sqrt{2}, 0)$.

**Claim:** $A_n = (n, n)$ and $B_n = (\sqrt{n(n + 1)}, 0)$ for all positive integers $n$

This will be proven by induction. The inductive hypothesis holds for the base case of $n = 1$. Assume that the hypothesis holds for $n = k$.

Then, $B_k = (\sqrt{k(k + 1)}, 0)$. The path from $B_k$ to $A_{k+1}$ is an upward facing parabola passing through $B_k$. Hence, $y = x^2 - k(k + 1)$. The point $A_{k+1}$ is the positive intersection of this parabola and $y = x$, implying that

$$x = x^2 - k(k + 1) \implies (x - (k + 1))(x + k) = 0 \implies x = k + 1.$$

Hence, $A_{k+1} = (k + 1, k + 1)$.

The path from $A_{k+1}$ to $B_{k+1}$ is a downward facing parabola passing through $A_{k+1}$. Hence, $y = -x^2 + (k + 1)(k + 2)$. The point $B_{k+1}$ is the positive intersection of this parabola and the $y$-axis, implying that

$$0 = -x^2 + (k + 1)(k + 2) \implies x = \sqrt{(k + 1)(k + 2)}.$$

Hence, $B_{k+1} = (\sqrt{(k + 1)(k + 2)}, 0)$, completing the induction.

It then follows that $B_n = (\sqrt{n(n+1)}, 0)$ for all positive integers $n$. Hence,

$$N = \sum_{n=1}^{20} n(n+1) = 2\sum_{n=1}^{20} \binom{n+1}{2} = 2 \cdot \binom{22}{3},$$

where the last step follows from the Hockey Stick Identity. Hence, $N = 3080$, for which the requested sum is $\boxed{\textbf{(E) } 11}$. ■