De Mathematics Competitions

1st Annual

DMC 12 A

Friday, May 27, 2022

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU DECIDE TO BEGIN.
2. This is a twenty-five question multiple choice test. For each question, only one answer choice is correct.
3. Mark your answer to each problem on the DMC 12 Answer Form with a keyboard. Check the keys for accuracy and erase errors and stray marks completely.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only blank scratch paper, rulers, compasses, and erasers are allowed as aids. Prohibited materials include calculators, smartwatches, phones, computing devices, protractors, and graph paper. No problems on the competition will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the competition, your non-existent proctor will not ask you to record certain information on the answer form.
8. You will have 75 minutes to complete the competition. You can discuss only with people that have already taken the competition in the private discussion forum until the end of the contest window.
9. When you finish the exam, don't sign your name in the space not provided on the Answer Form.

The DMC Committee reserves the right to disqualify scores from a school if it determines that the rules or the required security procedures were not followed.

The publication, reproduction or communication of the problems or solutions of this competition during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

Students who score well on this DMC 12 may or may not be invited to the 2023 DIME. More details about the DIME and other information are on the back page of this test booklet.

1. A red container is filled with water. If 20% of the water in the red container is poured into an empty blue container, what is the ratio of the amount of water in the blue container to the amount of water in the red container?

   (A) $1:6$ **(B)** $1:5$ **(C)** $1:4$ **(D)** $1:3$ **(E)** $1:2$

2. What is the smallest positive integer n such that

   $$\frac{1}{2}+\frac{1}{3}+\frac{1}{n}$$

   is less than 1?

   (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

3. A rectangle has an area of 22 square inches. If the height of the rectangle was 1 inch shorter, the rectangle would have an area of 18 square inches. What is the perimeter of the rectangle in inches?

   (A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 19

4. Let $f(x) = \log_9 x$, and let $g(x)$ be a function such that $g(x) = f(x^2)$ for all nonzero real values of x. Which of the following is equal to $g(x)$?

   (A) $\log_3 x$ **(B)** $\log_3 |x|$ **(C)** $2\log_3 |x|$ **(D)** $\log_9 \sqrt{x}$ **(E)** x^2

5. There are 10 students in a classroom with 12 chairs. Before lunch, each student sat on a different chair. After lunch, each student randomly chose a chair to sit on. If no two students chose the same chair after lunch, what is the probability that every chair had been sat on at least once?

   (A) $\frac{7}{24}$ **(B)** $\frac{11}{36}$ **(C)** $\frac{25}{66}$ **(D)** $\frac{14}{33}$ **(E)** $\frac{15}{22}$

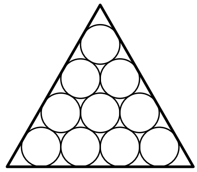
6. Bo writes down all the divisors of 144 on a board. He then erases some of the divisors. If no two of the divisors left on the board have a product divisible by 18, what is the least number of divisors Bo could have erased?

   (A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

7. Let a, b, and c be positive real numbers such that the ratio of a to bc is $1:3$, the ratio of b to ac is $1:12$, and the ratio of c to $a+b$ is $1:8$. What is $b+c$?

   (A) 22 **(B)** 24 **(C)** 26 **(D)** 28 **(E)** 30

8. In the figure below, 10 pairwise externally tangent circles of radius 1 are in a triangle, with the outer circles tangent to the sides of the triangle. Which of the following is closest to the area of the triangle?



(A) 36 **(B)** 39 **(C)** 42 **(D)** 45 **(E)** 48

9. In rectangle $ABCD$ with $AB = 5$ and $BC = 4$, let E be a point on side $\overline{CD}$. Given that segment $\overline{AE}$ bisects $\angle BED$, what is the length of $\overline{AE}$?

(A) $3\sqrt{2}$ **(B)** $2\sqrt{5}$ **(C)** $3\sqrt{3}$ **(D)** $2\sqrt{7}$ **(E)** $\sqrt{30}$

10. What is the value of

$$\frac{\sin(\frac{\pi}{24})\sin(\frac{2\pi}{24})\cdots\sin(\frac{5\pi}{24})}{\cos(\frac{6\pi}{24})\cos(\frac{7\pi}{24})\cdots\cos(\frac{11\pi}{24})}?$$

(A) 0 **(B)** $\frac{1}{2}$ **(C)** $\frac{\sqrt{2}}{2}$ **(D)** 1 **(E)** $\sqrt{2}$

11. Let a, b, and c be real numbers which satisfy

$$\begin{aligned} a + b + c &= 1, \\ a + |b| + |c| &= 4, \\ |a| + b + |c| &= 5, \\ |a| + |b| + c &= 8. \end{aligned}$$

What is $a^2 + b^2 + c^2$?

(A) $\frac{53}{2}$ **(B)** $\frac{55}{2}$ **(C)** $\frac{57}{2}$ **(D)** $\frac{59}{2}$ **(E)** $\frac{61}{2}$

12. Let p, q, r, and s be (not necessarily distinct) primes that satisfy

$$p = q^2 + r^2 + s^2 + qrs.$$

What is the sum of the digits of the second smallest possible value of p?

(A) 5 **(B)** 7 **(C)** 8 **(D)** 10 **(E)** 11

13. A positive integer $n > 1$ is called *toasty* if for all integers m with $1 \le m < n$, there exists a positive integer k such that

$$\frac{m}{n} = \frac{k}{k+12}.$$

How many toasty integers are there?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 6

14. Let $\lfloor r \rfloor$ denote the greatest integer less than or equal to a real number r. Let N be the number of positive integers $n \le 100$ such that

$$\lfloor (n+1)\pi \rfloor - \lfloor n\pi \rfloor = 4.$$

What is the sum of the digits of N?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

15. Let A be the real part of a randomly chosen solution to the equation $z^{27} = 1$, and let B be the real part of a randomly chosen solution to the equation $z^9 = 1$. What is the probability that $A > B$?

(A) $\frac{112}{243}$ **(B)** $\frac{38}{81}$ **(C)** $\frac{13}{27}$ **(D)** $\frac{119}{243}$ **(E)** $\frac{40}{81}$

16. How many sequences of the first 8 positive integers $a_1, a_2, \ldots, a_8$ are there such that $a_{2i-1} < a_{2i}$ for all odd i, $a_{2i-1} > a_{2i}$ for all even i, and the even integers within the sequence are listed in increasing order?

(A) 42 **(B)** 66 **(C)** 78 **(D)** 102 **(E)** 108

17. Let $ABCD$ be a trapezoid with $\overline{AB} \parallel \overline{CD}$, $AB < CD$, and $AD = BC = 5$. Let M be the midpoint of side $\overline{AD}$. Given that $BM = 6$, and the area of $ABCD$ is as large as possible, what is $AB + CD$?

(A) 11 **(B)** $\frac{45}{4}$ **(C)** $\frac{35}{3}$ **(D)** 13 **(E)** $\frac{66}{5}$

18. Each of the N students in Mr. Ji's class took a 10-question quiz with questions $1, 2, \ldots, 10$. Suppose for every (possibly empty) subset of $\{1, 2, \ldots, 10\}$, there exists a student who got exactly those questions correct, and for every $i = 0, 1, 2, \ldots, 10$, if a student got i questions correct, then of the students that got those same i questions correct (including that student), the fraction of them that got over i questions correct is $1 - 2^{i-10}$. If 3 students got a perfect score, what is the remainder when N is divided by 100?

(A) 24 **(B)** 32 **(C)** 48 **(D)** 64 **(E)** 72

19. In isosceles $\triangle ABC$ with $AB = AC = 4$ and $BC = 2$, let point D, distinct from B, be on side $\overline{AB}$ such that $CD = 2$. The circle passing through B, C, and D intersects side $\overline{AC}$ and the line through C perpendicular to $\overline{AB}$ at points P and Q, respectively, both distinct from C. If PQ^2 is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers, what is $m + n$?

(A) 45 **(B)** 46 **(C)** 63 **(D)** 64 **(E)** 71

20. Let $ABCD$ be a rectangle with $AB > BC$. Let E be a point on side $\overline{AD}$, and let $CEFG$ be the rectangle where B is on side $\overline{FG}$. Let H be the point on side $\overline{CD}$ such that $\overline{BH} \perp \overline{CE}$. If $CH = 2$, $DE = 3$, and the area of $CEFG$ is 48, then $AC = m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. What is $m + n$?

(A) 24 **(B)** 26 **(C)** 28 **(D)** 30 **(E)** 32

21. How many ordered pairs of positive integers (m, n) satisfy the following?

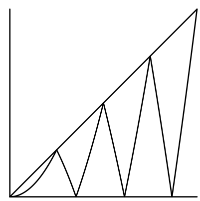
"There are exactly m set(s) of 100 consecutive positive integers whose least element is less than 100 which contain exactly $m + 1$ multiples of n."

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

22. An ellipse has foci A and B with $AB = 2$. There is a sequence of n distinct points $P_1, P_2, \ldots, P_n$ on the boundary of the ellipse such that $AP_i^2 + BP_i^2 = 4$ for all $1 \le i \le n$. What is the maximum possible area of $P_1P_2 \cdots P_n$?

(A) $\frac{1}{2}$ **(B)** $\frac{\sqrt{2}}{2}$ **(C)** 1 **(D)** $\sqrt{2}$ **(E)** 2

23. In the xy-plane, a laser emanates from the origin with a path whose shape obeys $y = x^2$. Whenever the laser touches the line $y = x$, the path of the laser will reflect over the line parallel to the x-axis passing though where the laser last touched $y = x$, and whenever the laser touches the x-axis, the path of the laser will reflect over the x-axis. The graph below shows the path of the laser and its first 7 reflection points. If N denotes the sum of the squares of the x-coordinates of the first 20 points where the laser intersects the x-axis (excluding the origin), what is the sum of the digits of N?



(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

24. A sequence $a_1, a_2, \ldots, a_{2022}$ is such that $a_i \in \{-2, -1, 0, 1, 2\}$ for each $1 \le i \le 2022$, and

$$\begin{aligned} a_1 + a_2 + a_3 + \cdots + a_{2022} &= 20, \\ a_1^2 + a_2^2 + a_3^2 + \cdots + a_{2022}^2 &= 100, \\ a_1^3 + a_2^3 + a_3^3 + \cdots + a_{2022}^3 &= 50. \end{aligned}$$

What is the minimum value of $a_1^4 + a_2^4 + a_3^4 + \cdots + a_{2022}^4$?

(A) 150 **(B)** 160 **(C)** 170 **(D)** 180 **(E)** 190

25. Eric randomly places 3 circles of radius 1 on a 7-by-7 grid of squares, each with side length 1, such that each circle is at least partially on the grid. Which of the following is closest to the average number of squares that are at least partially covered by a circle? (The circles may overlap.)

(A) 9 **(B)** 11 **(C)** 13 **(D)** 15 **(E)** 17

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DO NOT OPEN UNTIL FRIDAY, May 27, 2022

****Administration on an earlier date will disqualify your results.****

- All the information needed to administer this exam is not contained in the non-existent DMC 12 Teacher's Manual. PLEASE READ THE MANUAL BEFORE FRIDAY, MAY 27, 2022.
- Send **DeToasty3**, **HrishiP**, and **pog** a private message on Art of Problem Solving submitting your answers to the DMC 12. Alternatively, you may submit your answers via a Google Form linked in the opening post.
- The publication, reproduction or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.

For more information about the DMC and our other competitions, please visit `https://detoasty3.github.io/dmc.html`.

Questions and comments about this competition should be sent to:

DeToasty3.