De Mathematics Competitions

Official Solutions

De Mathematics Competitions

3rd Annual

DMC 12 A

Friday, May 27, 2022

This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

Questions and complaints about this competition should be sent by private message to

DeToasty3.

Answer Key:

1. (C)	2. (D)	3. (E)	4. (B)	5. (E)
6. (C)	7. (A)	8. (B)	9. (B)	10. (E)
11. (A)	12. (C)	13. (D)	14. (A)	15. (A)
16. (D)	17. (D)	18. (E)	19. (D)	20. (C)
21. (D)	22. (E)	23. (E)	24. (B)	25. (C)

Problem 1:

(pog) A red container is filled with water. If 20% of the water in the red container is poured into an empty blue container, what is the ratio of the amount of water in the blue container to the amount of water in the red container?

(A) $1:6$ **(B)** $1:5$ **(C)** $1:4$ **(D)** $1:3$ **(E)** $1:2$

Answer (C):

We get that $100\% - 20\% = 80\%$ of the water is in the red container and 20% of the water is in the blue container, so the requested ratio is $\boxed{\textbf{(C) } 1:4}$. ■

Problem 2:

(DeToasty3) What is the smallest positive integer n such that

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{n}$$

is less than 1?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Answer (D):

Computing, we get that $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$. Thus, we have that $\frac{5}{6}+\frac{1}{n}<1$, so $\frac{1}{n}<\frac{1}{6}$. Consequently, $n>6$, so the smallest positive integer value of n is $\boxed{\textbf{(D) } 7}$. ■

Problem 3:

(pog) A rectangle has an area of 22 square inches. If the height of the rectangle was 1 inch shorter, the rectangle would have an area of 18 square inches. What is the perimeter of the rectangle in inches?

(A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 19

Answer (E):

The width of the rectangle is $22 - 18 = 4$, so the height of the rectangle is $\frac{22}{4}$. Hence, the requested perimeter is $2(4 + \frac{22}{4}) = \boxed{\textbf{(E)}\ 19}$. ■

Problem 4:

(pog) Let $f(x) = \log_9 x$, and let $g(x)$ be a function such that $g(x) = f(x^2)$ for all nonzero real values of x. Which of the following is equal to $g(x)$?

(A) $\log_3 x$ **(B)** $\log_3 |x|$ **(C)** $2\log_3 |x|$ **(D)** $\log_9 \sqrt{x}$ **(E)** x^2

Answer (B):

Recall that $\log(x)$ is defined if and only if x is positive. Since x^2 is positive for all nonzero real values of x, $f(x^2)$ is defined for all nonzero real values of x. Hence, $g(x)$ must be defined for all nonzero real values of x, as well.

By the change of base rule,

$$f(x) = \log_9 x = \frac{\log(x)}{\log(9)} = \frac{\log(x)}{2\log(3)}.$$

If $x > 0$,

$$g(x) = f(x^2) = \frac{\log(x^2)}{2\log(3)} = \frac{2\log(|x|)}{2\log(3)} = \log_3(x).$$

If $x < 0$,

$$g(x) = f(x^2) = \frac{\log(x^2)}{2\log(3)} = \frac{2\log(|x|)}{2\log(3)} = \log_3(-x).$$

Hence, $g(x) = \boxed{\textbf{(B)}\ \log_3 |x|}$. ■

Problem 5:

(pog & DeToasty3) There are 10 students in a classroom with 12 chairs. Before lunch, each student sat on a different chair. After lunch, each student randomly chose a chair to sit on. If no two students chose the same chair after lunch, what is the probability that every chair had been sat on at least once?

(A) $\frac{7}{24}$ **(B)** $\frac{11}{36}$ **(C)** $\frac{25}{66}$ **(D)** $\frac{14}{33}$ **(E)** $\frac{15}{22}$

Answer (E):

Suppose the chairs are numbered $1, 2, \ldots, 12$. Without loss of generality, assume that the students sit in the chairs $1, 2, \ldots, 10$ before lunch. Then, for every chair to have been sat on at least once after lunch, 2 students must sit in chairs 11 and 12, while the remaining 8 each sit in one of the first 10 seats. There are $\binom{10}{8} = 45$ ways to determine such a seating. Since there are $\binom{12}{10} = 66$ possible seatings, the desired probability is $\dfrac{45}{66} = \boxed{\textbf{(E)}\ \dfrac{15}{22}}$. ■

Problem 6:

(DeToasty3) Bo writes down all the divisors of 144 on a board. He then erases some of the divisors. If no two of the divisors left on the board have a product divisible by 18, what is the least number of divisors Bo could have erased?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

Answer (A):

Clearly, Bo should erase all divisors on the board that are divisible by 18. This leaves $1, 2, 4, 8, 16, 3, 6, 12, 24, 48$, and 9 left on the board, accounting for 4 divisors erased so far.

Consider the set $S = \{3, 6, 12, 24, 48\}$. For any two distinct integers from S, at least one integer will be even. In addition, any integer in S is divisible by 3. Hence, the product of any two distinct integers in S is divisible by 18. Thus, at most 1 element of S can be left on the board, implying at least 4 more divisors will have to be erased.

If 9 is not erased, then $2, 4, 8$, and 16 must be erased. Thus, at least 1 more divisor will have to be erased.

Consequently, at least $4 + 4 + 1 = 9$ divisors must be erased. Indeed, if the divisors left on the board are 1, 2, 4, 8, 16, and 3, no two of these divisors have a product divisible by 18 because there is only one integer on the board divisible by 3 and that integer is not divisible by 9. Hence, the requested minimum is $\boxed{\textbf{(C)}\ 9}$. ■

Problem 7:

(DeToasty3 & pog) Let a, b, and c be positive real numbers such that the ratio of a to bc is $1:3$, the ratio of b to ac is $1:12$, and the ratio of c to $a+b$ is $1:8$. What is $b+c$?

(A) 22 **(B)** 24 **(C)** 26 **(D)** 28 **(E)** 30

Answer (A):

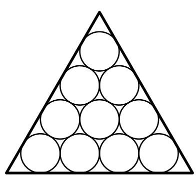
We get that $bc = 3a$ and $ac = 12b$. Hence,

$$bc \cdot ac = abc^2 = 3a \cdot 12b = 36ab,$$

so thus $c^2 = 36$. Since c is positive, we get that $c = 6$. Thus, substituting $c = 6$ into our three ratios, we have that $6b = 3a$, $6a = 12b$, and $a+b = 48$. Solving, we get that $a = 32$ and $b = 16$. Consequently, $b + c = 16 + 6 = \boxed{\text{(A) } 22}$. ■

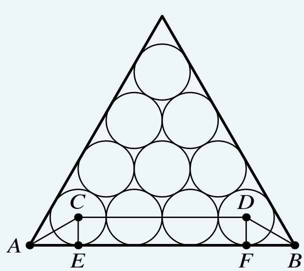
Problem 8:

(pog) In the figure below, 10 pairwise externally tangent circles of radius 1 are in a triangle, with the outer circles tangent to the sides of the triangle. Which of the following is closest to the area of the triangle?



(A) 36 **(B)** 39 **(C)** 42 **(D)** 45 **(E)** 48

Answer (B):



Observe that $CDFE$ is a rectangle and $EF = CD = 1 + 2 + 2 + 1 = 6$. In addition, $\triangle ACE$ and $\triangle BDF$ are 30-60-90 triangles. Since $CE = DF = 1$, $AE = FB = \sqrt{3}$. Hence, $AB = AE + EF + FB = 6 + 2\sqrt{3}$.

Then, $[ABC] = \frac{AB^2\sqrt{3}}{4} = 12\sqrt{3} + 18$. Since $\sqrt{3} \approx 1.73$, we get that $12\sqrt{3} + 18 \approx 38.76 \approx \boxed{\textbf{(B) } 39}$. ■

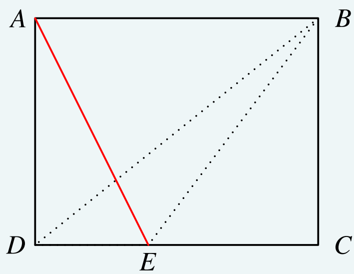
Problem 9:

(DeToasty3) In rectangle $ABCD$ with $AB = 5$ and $BC = 4$, let E be a point on side $\overline{CD}$. Given that segment $\overline{AE}$ bisects $\angle BED$, what is the length of $\overline{AE}$?

(A) $3\sqrt{2}$ **(B)** $2\sqrt{5}$ **(C)** $3\sqrt{3}$ **(D)** $2\sqrt{7}$ **(E)** $\sqrt{30}$

Answer (A):

Consider the following diagram:



We know that $\overline{AB} \parallel \overline{CD}$ and that $\angle BEA = \angle DEA$, so $\angle BEA = \angle BAE$. Consequently, $AB = BE = 5$. Since $BE = 5$, $BC = 4$, and $\angle BCE = 90°$, we have that $EC = 3$, which implies that $DE = 5 - EC = 2$. Therefore,

$$AE = \sqrt{AD^2 + DE^2} = \sqrt{16 + 4} = \sqrt{20} = \boxed{\textbf{(B) } 2\sqrt{5}}.$$

■

Problem 10:

(dc495 & pog) What is the value of

$$\frac{\sin(\frac{\pi}{24})\sin(\frac{2\pi}{24})\cdots\sin(\frac{5\pi}{24})}{\cos(\frac{6\pi}{24})\cos(\frac{7\pi}{24})\cdots\cos(\frac{11\pi}{24})}?$$

(A) 0 **(B)** $\frac{1}{2}$ **(C)** $\frac{\sqrt{2}}{2}$ **(D)** 1 **(E)** $\sqrt{2}$

Answer (E):

Consider the identity $\sin(\theta) = \cos(\frac{\pi}{2} - \theta)$. It then follows that

$$\cos(\tfrac{7\pi}{24})\cos(\tfrac{8\pi}{24})\cdots\cos(\tfrac{11\pi}{24}) = \sin(\tfrac{\pi}{24})\sin(\tfrac{2\pi}{24})\cdots\sin(\tfrac{5\pi}{24}).$$

Hence,

$$\frac{\sin(\frac{\pi}{24})\sin(\frac{2\pi}{24})\cdots\sin(\frac{5\pi}{24})}{\cos(\frac{6\pi}{24})\cos(\frac{7\pi}{24})\cdots\cos(\frac{11\pi}{24})} = \frac{\sin(\frac{\pi}{24})\sin(\frac{2\pi}{24})\cdots\sin(\frac{5\pi}{24})}{\cos(\frac{6\pi}{24})\sin(\frac{\pi}{24})\sin(\frac{2\pi}{24})\cdots\sin(\frac{5\pi}{24})},$$

so our answer is $\frac{1}{\cos(\frac{\pi}{4})} = \boxed{\textbf{(E)}\ \sqrt{2}}$. ■

Problem 11:

(pog) Let a, b, and c be real numbers which satisfy

$$\begin{aligned} a + b + c &= 1, \\ a + |b| + |c| &= 4, \\ |a| + b + |c| &= 5, \\ |a| + |b| + c &= 8. \end{aligned}$$

What is $a^2 + b^2 + c^2$?

(A) $\dfrac{53}{2}$ **(B)** $\dfrac{55}{2}$ **(C)** $\dfrac{57}{2}$ **(D)** $\dfrac{59}{2}$ **(E)** $\dfrac{61}{2}$

Answer (A):

Taking advantage of symmetry, we add the equations together, giving

$$2(a + b + c + |a| + |b| + |c|) = 18.$$

Thus, $|a| + |b| + |c| = 8$. We get that $|a| - a = 4$ and $|b| - b = 3$, so thus $(a, b) = \left(-2, -\frac{3}{2}\right)$. Since $a + b + c = 1$, we get that $c = \frac{9}{2}$, so our answer is

$$(-2)^2 + \left(-\frac{3}{2}\right)^2 + \left(\frac{9}{2}\right)^2 = 4 + \frac{9}{4} + \frac{81}{4} = \frac{106}{4} = \boxed{\textbf{(A)}\ \frac{53}{2}}.$$

■

Problem 12:

(HrishiP) Let p, q, r, and s be (not necessarily distinct) primes that satisfy

$$p = q^2 + r^2 + s^2 + qrs.$$

What is the sum of the digits of the second smallest possible value of p?

(A) 5 **(B)** 7 **(C)** 8 **(D)** 10 **(E)** 11

Answer (C):

Since q, r, and s are greater than or equal to 2, we get that

$$p \geq q^2 + r^2 + s^2 \geq 8 > 2.$$

Hence, p cannot be 2, and thus, must be an odd prime. If all of q, r, and s are odd, then the right hand side of the given equation is even, which is a contradiction. Hence, one of q, r, or s is 2. Since q, r, and s and interchangeable, without loss of generality, assume that $q = 2$. Then,

$$p = r^2 + s^2 + 2rs + 4.$$

If both r and s are odd, then the right hand side of this equation is even, which is a contradiction. Hence, one of r or s is 2. Without loss of generality, assume that $r = 2$. Then,

$$p = s^2 + 4s + 8 = (s + 2)^2 + 4.$$

Considering the two smallest odd primes for s, $s = 3$ implies $p = 29$, while $s = 5$ implies $p = 53$, both of which are prime. Hence, the second smallest possible value of p is 53. The requested sum is $\boxed{\textbf{(C) } 8}$. ■

Problem 13:

(DeToasty3) A positive integer $n > 1$ is called *toasty* if for all integers m with $1 \leq m < n$, there exists a positive integer k such that

$$\frac{m}{n} = \frac{k}{k + 12}.$$

How many toasty integers are there?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 6

Answer (D):

For a given pair of positive integers (m, n),

$$\frac{m}{n} = \frac{k}{k+12} \implies m(k+12) = nk \implies k(n-m) = 12m \implies k = \frac{12m}{n-m}.$$

Hence, for k to be an integer, $12m$ must be divisible by $n - m$.

For a positive integer $n \geq 2$ to be toasty, it is necessary for the divisibility to hold for $m = 1$. Hence, $n - 1$ divides 12, implying that the only n that could be toasty are $2, 3, 4, 5, 7$, and 13.

By the definition of a toasty integer, for $n = 2$ to be toasty, it sufficient for the divisibility to hold for $m = 1$. Since the divisibility holds, $n = 2$ is toasty.

Assume an integer $n \geq 3$ is toasty. Then, it is necessary for the divisibility to also hold for $m = 2$, implying that $n - 2$ divides 24. It then follows that 7 and 13 cannot be toasty.

Checking the other integers, $n = 3$ is toasty, since the divisibility holds for both $m = 1$ and $m = 2$. Similarly, $n = 4$ and $n = 5$ are toasty, since it can be verified that the divisibility holds for all integers m such that $1 \leq m < n$. Hence, all the toasty integers are 2, 3, 4, and 5, implying there are $\boxed{\textbf{(D) } 4}$ toasty integers. ■

Problem 14:

(DeToasty3) Let $\lfloor r \rfloor$ denote the greatest integer less than or equal to a real number r. Let N be the number of positive integers $n \leq 100$ such that

$$\lfloor (n+1)\pi \rfloor - \lfloor n\pi \rfloor = 4.$$

What is the sum of the digits of N?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Answer (A):

For every positive integer n, $(n+1)\pi$ and $n\pi$ are real numbers that differ by π. Since $3 < \pi < 4$, it follows that $\lfloor (n+1)\pi \rfloor$ and $\lfloor n\pi \rfloor$ differ by either 3 or 4.

Let a be the number of positive integers n such that $1 \leq n \leq 100$ and $\lfloor (n+1)\pi \rfloor - \lfloor n\pi \rfloor = 3$. Define b similarly for $\lfloor (n+1)\pi \rfloor - \lfloor n\pi \rfloor = 4$. Clearly,

$a + b = 100$.

Observe that

$$\lfloor 101\pi \rfloor - \lfloor \pi \rfloor = (\lfloor 2\pi \rfloor - \lfloor \pi \rfloor) + (\lfloor 3\pi \rfloor - \lfloor 2\pi \rfloor) + \ldots + (\lfloor 101\pi \rfloor - \lfloor 100\pi \rfloor) = 3a + 4b.$$

Using $\pi \approx 3.1415$, we get that $\lfloor 101\pi \rfloor = 317$ and $\lfloor \pi \rfloor = 3$, so $3a + 4b = 314$. In combination with $a + b = 100$, this implies that $(a, b) = (86, 14)$. It then follows that $N = 14$, for which the requested sum is $\boxed{\textbf{(A) } 5}$. ■

Problem 15:

(pog) Let A be the real part of a randomly chosen solution to the equation $z^{27} = 1$, and let B be the real part of a randomly chosen solution to the equation $z^9 = 1$. What is the probability that $A > B$?

(A) $\frac{112}{243}$ **(B)** $\frac{38}{81}$ **(C)** $\frac{13}{27}$ **(D)** $\frac{119}{243}$ **(E)** $\frac{40}{81}$

Answer (A):

Note that $B_n = A_{3n}$. Since the graph of $\cos(x)$ is symmetrical over $x = \pi$, we get that $A_n = A_{27-n}$ and $B_n = B_{9-n}$.

Case 1: A_0 is chosen

Then there are 8 ordered pairs: $(A_0, B_1), (A_0, B_2), \ldots, (A_0, B_8)$.

Case 2: B_0 is chosen

Since A is at most 1, there are no possible values since $B_0 = 1$.

Case 3: $1 > A_n > B$

If B_1 or B_8 is chosen, then $1 \leq n \leq 2$, and each A_n corresponds to another A_n across $x = \pi$, for 4 cases.

If B_2 or B_7 is chosen, then $1 \leq n \leq 5$, and each A_n corresponds to another A_n across $x = \pi$, for 10 cases.

If B_3 or B_6 is chosen, then $1 \leq n \leq 8$, and each A_n corresponds to another A_n across $x = \pi$, for 16 cases.

If B_4 or B_5 is chosen, then $1 \leq n \leq 11$, and each A_n corresponds to another A_n across $x = \pi$, for 22 cases.

Hence, there are $2(4+10+16+22) = 104$ ordered pairs such that $1 > A_n > B$.

There are $27 \cdot 9$ choices total, so hence the requested probability is $\frac{8+104}{27 \cdot 9} = \boxed{\textbf{(A)}\ \frac{112}{243}}$. ■

Problem 16:

(DeToasty3) How many sequences of the first 8 positive integers $a_1, a_2, \ldots, a_8$ are there such that $a_{2i-1} < a_{2i}$ for all odd i, $a_{2i-1} > a_{2i}$ for all even i, and the even integers within the sequence are listed in increasing order?

(A) 42 **(B)** 66 **(C)** 78 **(D)** 102 **(E)** 108

Answer (D):

Split the sequence into four pairs (a_1, a_2), (a_3, a_4), (a_5, a_6), and (a_7, a_8) and call them pairs 1 through 4. We see that once we choose the two numbers to go in each pair, the orders are fixed. In order for the evens to be listed in increasing order, we must either have all evens be in different pairs, or if two evens are in the same pair, they must be in pairs 1 or 3, since those are increasing. We have a few cases:

Case 1: Evens are in different pairs.

Then, there is one way to place the evens given $4! = 24$ ways to place the odds.

Case 2: Evens are all in pairs 1 and 3.

Then, 2 and 4 must be in pair 1, and 6 and 8 must be in pair 3. Then, there are $\binom{4}{2} = 6$ ways to place the odds in the remaining two empty pairs.

Case 3: Pair 1 has two evens, but not pair 3.

Then, 2 and 4 must be in pair 1, and 6 and 8 must be in different pairs. There are 3 ways to choose which pairs 6 and 8 go into. Then, there are 4 ways to choose which odd gets paired with 6, 3 ways to choose which odd gets paired with 8, and the other two get placed in the remaining empty pair. This gives us $3 \cdot 4 \cdot 3 = 36$ ways.

Case 4: Pair 3 has two evens, but not pair 1.

Then, we either have that pair 3 contains 6 and 8, or it contains 4 and 6. If pair 3 contains 6 and 8, then 2 must be in pair 1, and 4 must be in pair 2. Then, there are 4 ways to choose which odd gets paired with 2, 3 ways to choose which odd gets paired with 4, and the other two get placed in the remaining empty pair. If pair 3 contains 4 and 6, then 2 can be in either pairs 1 or 2, and 8 must be in pair 4. Then, there are 4 ways to choose which odd gets paired with 2, 3 ways to choose which odd gets paired with 8, and the other two get placed in the remaining empty pair. This gives us $4 \cdot 3 + 2 \cdot 4 \cdot 3 = 36$ ways.

Thus, our answer is $24 + 6 + 36 + 36 = \boxed{\textbf{(D) } 102}$. ■

Problem 17:

(DeToasty3) Let $ABCD$ be a trapezoid with $\overline{AB} \parallel \overline{CD}$, $AB < CD$, and $AD = BC = 5$. Let M be the midpoint of side $\overline{AD}$. Given that $BM = 6$, and the area of $ABCD$ is as large as possible, what is $AB + CD$?

(A) 11 **(B)** $\frac{45}{4}$ **(C)** $\frac{35}{3}$ **(D)** 13 **(E)** $\frac{66}{5}$

Answer (D):

Extend BM and CD to intersect at E. Clearly, $\angle BMA = \angle EMD$. In addition, by the parallel lines, $\angle ABM = \angle DEM$. Hence, $\triangle ABM \sim \triangle DEM$. Furthermore, since $AM = MD$, $\triangle ABM$ and $\triangle DEM$ are congruent. This implies that $EM = 6$.
Since $\triangle ABM \cong \triangle DEM$, $[ABM] = [DEM]$. Furthermore,

$$[ABCD] = [ABM] + [BCDM] = [DEM] + [BCDM] = [BCE].$$

Hence, to maximize $[ABCD]$, it suffices to maximize $[BCE]$. Since BE and BC are fixed lengths, $[BCE]$ is maximized when $\angle CBE = 90^\circ$. Then,

$$AB + CD = DE + CD = CE.$$

By Pythagorean Theorem on $\triangle BCE$, our answer is $CE = \boxed{\textbf{(D) } 13}$. ■

Problem 18:

(DeToasty3 & HrishiP) Each of the N students in Mr. Ji's class took a 10-question quiz with questions $1, 2, \ldots, 10$. Suppose for every (possibly empty) subset of $\{1, 2, \ldots, 10\}$, there exists a student who got exactly those questions correct, and for every $i = 0, 1, 2, \ldots, 10$, if a student got i questions correct, then of the students that got those same i questions correct (including that student), the fraction of them that got over i questions correct is $1 - 2^{i-10}$. If 3 students got a perfect score, what is the remainder when N is divided by 100?

(A) 24 **(B)** 32 **(C)** 48 **(D)** 64 **(E)** 72

Answer (E):

I claim that the number of students who got exactly some (possibly empty) subset of questions $\{1, 2, \ldots, 10\}$ correct is 3. We will use induction, where we suppose that 3 students got a subset of n problems correct, for some $1 \le n \le 10$. Our base case is $n = 10$. Then, since we are given that 3 students got a perfect score, we are done. Next, without loss of generality, suppose that 3 students got exactly questions $1, 2, \ldots, n$ correct, for $n = 10, 9, \ldots, k + 1$. We will show that 3 students got exactly questions $1, 2, \ldots, k$ correct. Note that for each non-empty subset of questions $\{k + 1, k + 2, \ldots, 10\}$, exactly 3 students got all of questions $1, 2, \ldots, k$ correct as well as the subset of questions $\{k + 1, k + 2, \ldots, 10\}$ correct. Since there are $2^{10-k} - 1$ subsets, there are $3 \cdot 2^{10-k} - 3$ students who got questions $1, 2, \ldots, k$ as well as other questions correct. Then, we have from the problem statement that if S students got at least questions $1, 2, \ldots, k$ correct, then

$$\frac{3 \cdot 2^{10-k} - 3}{S} = 1 - 2^{k-10} \implies S = 3 \cdot 2^{10-k}.$$

Thus, we have that $S - (3 \cdot 2^{10-k} - 3) = 3$ students got exactly questions $1, 2, \ldots, k$ correct. Thus,

$$N = 3\left(\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \cdots + \binom{10}{10}\right) = 3 \cdot 1024 = 3072.$$

Thus, the answer is $\boxed{\textbf{(E) } 72}$. ■

Problem 19:

(HrishiP & DeToasty3) In isosceles $\triangle ABC$ with $AB = AC = 4$ and $BC = 2$, let point D, distinct from B, be on side $\overline{AB}$ such that $CD = 2$. The circle passing through B, C, and D intersects side $\overline{AC}$ and the line through C perpendicular to $\overline{AB}$ at points P and Q, respectively, both distinct from C. If PQ^2 is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers, what is $m + n$?

(A) 45 **(B)** 46 **(C)** 63 **(D)** 64 **(E)** 71

Answer (D):

Let ω denote the circle passing through B, C, and D. In addition, let M and N be the midpoints of BC and BD, respectively. By the symmetry of $\triangle ABC$ across AM, DP is parallel to BC, implying that $BCPD$ is an isosceles trapezoid. In addition, by the symmetry of $\triangle BCD$ across CN, CQ is a diameter of ω. Hence, $\angle CBQ = \angle CPQ = 90^\circ$.

Clearly, $\angle BAM$ and $\angle BCN$ are complementary to $\angle ABC$. Hence, $\angle BAM = \angle QCB$. In addition, $\angle AMB = \angle CBQ = 90^\circ$, implying that $\triangle AMB \sim \triangle CBQ$. Observe that $BM = 1$. By Pythagorean Theorem on $\triangle ABM$, we have that $AM = \sqrt{15}$.

Using $\triangle AMB \sim \triangle CBQ$,

$$\frac{CQ}{CB} = \frac{AB}{AM} = \frac{4}{\sqrt{15}} \implies CQ = \frac{8}{\sqrt{15}}.$$

Observe that $\triangle ABC$ and $\triangle CBD$ are both isosceles and share $\angle ABC$. Since $\angle CDB = \angle CBD = \angle ABC = \angle ACB$, $\triangle ABC \sim \triangle CBD$. Hence,

$$\frac{BD}{BC} = \frac{BC}{AB} \implies BD = 1.$$

By Pythagorean Theorem on $\triangle PQC$, $PQ^2 = \frac{49}{15}$. The requested sum is $\boxed{\textbf{(D) } 64}$. ■

Problem 20:

(DeToasty3) Let $ABCD$ be a rectangle with $AB > BC$. Let E be a point on side $\overline{AD}$, and let $CEFG$ be the rectangle where B is on side $\overline{FG}$. Let H be the point on side $\overline{CD}$ such that $\overline{BH} \perp \overline{CE}$. If $CH = 2$, $DE = 3$, and the area of $CEFG$ is 48, then $AC = m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. What is $m + n$?

(A) 24 **(B)** 26 **(C)** 28 **(D)** 30 **(E)** 32

Answer (C):

Let CE and BH intersect at I. By right triangles $\triangle BCH$ and $\triangle CIH$, $\angle CBH$ and $\angle BHC$ are complementary and $\angle IHC$ and $\angle ICH$ are complementary. Hence, $\angle CBH = \angle DCE$. Since $\angle BCH = \angle CDE = 90^\circ$, $\triangle BCH \sim \triangle CDE$. Hence, $BC : CD = CH : DE = 2 : 3$.

Observe that $\triangle BCE$ shares the same base and height as rectangle $CEFG$. Hence, $[BCE] = \frac{1}{2}[CEFG]$. Similarly, $\triangle BCE$ shares the same base and height as rectangle $ABCD$, implying that $[BCE] = \frac{1}{2}[ABCD]$. Hence, $ABCD$ and $CEFG$ have the same area, implying that $[ABCD] = 48$.

Since $BC : CD = 2 : 3$, $BC = 4\sqrt{2}$ and $AB = CD = 6\sqrt{2}$. By Pythagorean Theorem, $AC = 2\sqrt{26}$. The requested sum is $\boxed{\textbf{(C) } 28}$. ■

Problem 21:

(DeToasty3) How many ordered pairs of positive integers (m, n) satisfy the following?
"There are exactly m set(s) of 100 consecutive positive integers whose least element is less than 100 which contain exactly $m + 1$ multiples of n."

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Answer (D):

We have two cases for what n is: $n < 100$ and $n \geq 100$. If $n < 100$, then consider the set of 100 consecutive positive integers whose smallest element is a multiple of n. Then, we have that there are $\lfloor \frac{99}{n} \rfloor + 1$ multiples of n in this set because after the smallest element is a multiple of n, we have 99 more elements starting from $1 \pmod{n}$, so we will hit $\lfloor \frac{99}{n} \rfloor$ more multiples of n. There are also $\lfloor \frac{99}{n} \rfloor$ multiples of n between 1 and 99, inclusive.

If n is not a divisor of 99, then for each multiple of n between 1 and 99, inclusive, we start with this multiple of n as the smallest element of a set and then repeatedly subtract each element by 1 until we get to a point where either the largest element is a multiple of n or where the smallest element is equal to 1. Thus, for each multiple of n between 1 and 99, inclusive, we can always generate more sets (provided that $n > 1$).

If $n \geq 100$, then there can be at most 1 multiple of n in a set. This is clearly not possible as m would be either -1 or 0, neither of which are positive. Thus, we require that n is a divisor of 99. Note that $99 = 3^2 \cdot 11$, which has $(2 + 1)(1 + 1) = 6$ divisors, so the answer is $\boxed{\textbf{(D) } 6}$. ■

Problem 22:

(HrishiP) An ellipse has foci A and B with $AB = 2$. There is a sequence of n distinct points $P_1, P_2, \ldots, P_n$ on the boundary of the ellipse such that $AP_i^2 + BP_i^2 = 4$ for all $1 \leq i \leq n$. What is the maximum possible area of $P_1 P_2 \cdots P_n$?

(A) $\frac{1}{2}$ **(B)** $\frac{\sqrt{2}}{2}$ **(C)** 1 **(D)** $\sqrt{2}$ **(E)** 2

Answer (E):

Since P_i is a point that satisfies $AP_i^2 + BP_i^2 = AB^2$, we get that $\angle AP_iB = 90^\circ$ by the converse of Pythagorean Theorem. Let ω be the circle with diameter AB, which consists of the only points P that can satisfy $\angle APB = 90^\circ$. Hence, all of $P_1, P_2, \ldots, P_n$ must lie on the circumference of ω.

By symmetry across AB, ω and the ellipse will intersect at an even number of points. In addition, an ellipse and circle can have at most 4 intersection points. If the two conic sections intersect at 2 points, P_1P_2 will not be a polygon. Thus, they intersect at exactly 4 points. By symmetry across the axes of the ellipse, the intersection points will be a rectangle. Since the radius of ω is fixed, the area of the quadrilateral will be maximized when the intersection points form a square.

Since the diameter of ω is 2, the side length of the square inscribed in ω is $\sqrt{2}$. Hence, the desired area is $\boxed{\textbf{(E) } 2}$. ■

Problem 23:

(HrishiP & DeToasty3) In the xy-plane, a laser emanates from the origin with a path whose shape obeys $y = x^2$. Whenever the laser touches the line $y = x$, the path of the laser will reflect over the line parallel to the x-axis passing though where the laser last touched $y = x$, and whenever the laser touches the x-axis, the path of the laser will reflect over the x-axis. The graph below shows the path of the laser and its first 7 reflection points. If N denotes the sum of the squares of the x-coordinates of the first 20 points where the laser intersects the x-axis (excluding the origin), what is the sum of the digits of N?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

Answer (E):

Let A_n denote the nth point where the laser touches $y = x$ after the origin. Similarly, let B_n denote the nth point where the laser touches the x-axis after the origin.

When the laser is reflected across a line parallel to the x-axis, the resulting parabola formed by the laser's path will still have its vertex on the y-axis. The absolute value of the x^2 coefficient will not change after the reflection, but the sign of the x^2 coefficient will change. Hence, the laser's path from A_n to B_n or B_n to A_{n+1} for all positive integers n can always be represented as the portion of the graph of $y = \pm x^2 + r$ for some real number r.

Clearly, $A_1 = (1, 1)$. The laser's path from A_1 to B_1 is a downward facing parabola that passes through $(1, 1)$, implying $y = -x^2 + 2$. This implies that $B_1 = (\sqrt{2}, 0)$.

Claim: $A_n = (n, n)$ and $B_n = (\sqrt{n(n+1)}, 0)$ for all positive integers n

This will be proven by induction. The inductive hypothesis holds for the base case of $n = 1$. Assume that the hypothesis holds for $n = k$.

Then, $B_k = (\sqrt{k(k+1)}, 0)$. The path from B_k to A_{k+1} is an upward facing parabola passing through B_k. Hence, $y = x^2 - k(k+1)$. The point A_{k+1} is the positive intersection of this parabola and $y = x$, implying that

$$x = x^2 - k(k+1) \implies (x - (k+1))(x + k) = 0 \implies x = k + 1.$$

Hence, $A_{k+1} = (k + 1, k + 1)$.

The path from A_{k+1} to B_{k+1} is a downward facing parabola passing through A_{k+1}. Hence, $y = -x^2 + (k+1)(k+2)$. The point B_{k+1} is the positive intersection of this parabola and the y-axis, implying that

$$0 = -x^2 + (k+1)(k+2) \implies x = \sqrt{(k+1)(k+2)}.$$

Hence, $B_{k+1} = (\sqrt{(k+1)(k+2)}, 0)$, completing the induction.

It then follows that $B_n = (\sqrt{n(n+1)}, 0)$ for all positive integers n. Hence,

$$N = \sum_{n=1}^{20} n(n+1) = 2\sum_{n=1}^{20} \binom{n+1}{2} = 2 \cdot \binom{22}{3},$$

where the last step follows from the Hockey Stick Identity. Hence, $N = 3080$, for which the requested sum is $\boxed{\textbf{(E) } 11}$. ■

Problem 24:

(Awesome_guy) A sequence $a_1, a_2, \dots, a_{2022}$ is such that $a_i \in \{-2, -1, 0, 1, 2\}$ for each $1 \le i \le 2022$, and

$$\begin{aligned} a_1 + a_2 + a_3 + \cdots + a_{2022} &= 20, \\ a_1^2 + a_2^2 + a_3^2 + \cdots + a_{2022}^2 &= 100, \\ a_1^3 + a_2^3 + a_3^3 + \cdots + a_{2022}^3 &= 50. \end{aligned}$$

What is the minimum value of $a_1^4 + a_2^4 + a_3^4 + \cdots + a_{2022}^4$?

(A) 150 **(B)** 160 **(C)** 170 **(D)** 180 **(E)** 190

Answer (B):

Let w denote the number of a_i that equal 2. Define x, y, and z similarly for 1, -1, and -2, respectively. For a sequence $a_1, a_2, \dots, a_{2022}$ to exist, $w + x + y + z \le 2022$. By the three given equations,

$$a_1 + a_2 + a_3 + \cdots + a_{2022} = 20 \implies 2w + x - y - 2z = 20 \quad (1)$$

$$a_1^2 + a_2^2 + a_3^2 + \cdots + a_{2022}^2 = 100 \implies 4w + x + y + 4z = 100 \quad (2)$$

$$a_1^3 + a_2^3 + a_3^3 + \cdots + a_{2022}^3 = 50 \implies 8w + x - y - 8z = 50. \quad (3)$$

In addition,

$$a_1^4 + a_2^4 + a_3^4 + \cdots + a_{2022}^4 = 16(w + z) + (x + y). \quad (4)$$

By considering (1) and (3) as a system of two equations in $w - z$ and $x - y$, $w - z = 5$ and $x - y = 10$. Since w, x, y, and z are nonnegative integers, $w + z \ge w - z$ and $x + y \ge x - y$. In addition, $w + z$ and $w - z$ have the same parity, and $x + y$ and $x - y$ have the same parity. Thus, there exists a nonnegative integer m such that $w + z = 5 + 2m$, and there exists a nonnegative integer n such that $x + y = 10 + 2n$. Substituting these into (2) and (4),

$$4(w + z) + (x + y) = 100 \implies 4m + n = 35$$

$$16(w+z)+(x+y) = 90+32m+2n = 90+2(4m+n)+24m = 90+2\cdot 35+24m.$$

To minimize this expression, it suffices to take $(m, n) = (0, 35)$, in which the minimum value is $\boxed{\textbf{(B) } 160}$.

Remark: Indeed, such a sequence exists. If $(m, n) = (0, 35)$, then $(w, x, y, z) = (5, 45, 35, 0)$, which satisfies $w + x + y + z \leq 2022$. ■

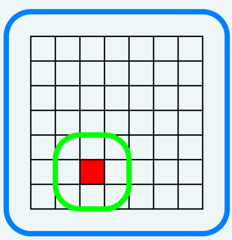
Problem 25:

(HrishiP) Eric randomly places 3 circles of radius 1 on a 7-by-7 grid of squares, each with side length 1, such that each circle is at least partially on the grid. Which of the following is closest to the average number of squares that are at least partially covered by a circle? (The circles may overlap.)

(A) 9 **(B)** 11 **(C)** 13 **(D)** 15 **(E)** 17

Answer (C):

Consider the locus of points for which a circle of radius 1 centered at the point overlaps at least partially with the grid. This is the set of all points that are a distance of 1 or less from some point on the grid. This locus is the region bounded by the blue curve in the diagram below.



The blue curve bounds 77 unit squares and 4 quarter sectors of a circle of radius 1.

For an arbitrary square of the grid, consider the locus of points for which a circle of radius 1 centered at the point overlaps at least partially with the square. One such square is highlighted in red in the diagram above. The locus of such points is the region bounded in green. The green curve bounds 5 unit squares and 4 quarter sectors of a circle of radius 1. In addition, for each square of the grid, its corresponding green curve lies completely within the blue curve.

Hence, for any given square of the grid, the probability that an arbitrarily placed circle of radius 1 at least partially overlaps with it is $\frac{5+\pi}{77+\pi}$. Thus, the probability that the circle does not overlap with the square is $\frac{72}{77+\pi}$. Consider the complement: the expected number of squares that do not have any of the 3 circles partially overlapping with it.

The probability that any given square does not have any of the 3 circles partially overlapping with it is $(\frac{72}{77+\pi})^3$. Hence, the expected number of such squares is $49 \cdot (\frac{72}{77+\pi})^3$. Since $3.141 < \pi < 3.142$, there exists a positive real number ϵ such that $\pi = 3 + \epsilon$ and $0.141 < \epsilon < 0.142$. Then,

$$49 \cdot \left(\frac{72}{77+\pi}\right)^3 = 49 \cdot \left(\frac{72}{80+\epsilon}\right)^3.$$

The size of ϵ is negligible compared to 80, and hence, may be ignored when approximating. Thus,

$$49 \cdot \left(\frac{72}{80+\epsilon}\right)^3 \approx 49 \cdot \left(\frac{72}{80}\right)^3 = 49 \cdot \frac{729}{1000} = 35.721 \approx 36.$$

Since this is complement, the expected number of squares that will be partially covered by a circle is approximately $\boxed{\textbf{(C) } 13}$. ■