Official Solutions

De Mathematics Competitions

3rd Annual

DMC 12 B

Friday, October 28, 2022

This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

Questions and complaints about this competition should be sent by private message to

DeToasty3.

(Also, special thanks to bissue, GammaZero, Oxymoronic15, & richy for contributing problems, and to P_Groudon & peace09 for helping with this solutions booklet!)

Answer Key:

1. (D)	2. (A)	3. (C)	4. (E)	5. (D)
6. (E)	7. (E)	8. (D)	9. (A)	10. (E)
11. (E)	12. (E)	13. (C)	14. (C)	15. (B)
16. (D)	17. (D)	18. (B)	19. (D)	20. (B)
21. (E)	22. (B)	23. (D)	24. (A)	25. (C)

Problem 1:

(Oxymoronic15) What is the value of $\frac{2022! \cdot 2019!}{2020! \cdot 2021!}$?

(A) $\frac{1009}{1010}$ **(B)** $\frac{2020}{2021}$ **(C)** $\frac{2021}{2022}$ **(D)** $\frac{1011}{1010}$ **(E)** $\frac{2023}{2022}$

Answer (D):

The given expression is equal to $\frac{2022!}{2021!} \cdot \frac{2019!}{2020!}$. Note that $\frac{2022!}{2021!} = 2022$ and $\frac{2019!}{2020!} = \frac{1}{2020}$, so our answer is $2022 \cdot \frac{1}{2020} = \frac{2022}{2020} = \boxed{\textbf{(D)}\ \frac{1011}{1010}}$. ■

Problem 2:

(pog) Taiki and Richard are playing frisbee. Each round, players earn 3 points for winning and 1 point for a tie. The game ends when someone gets 20 points. At the end of the game, Richard had lost 3 more rounds than Taiki. How many points did Richard have at the end of the game?

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 16

Answer (A):

For every round that Richard loses, he gets 3 points less than Taiki. Thus, Richard has $20 - 3 \cdot 3 = \boxed{\textbf{(A)}\ 11}$ points. ■

Problem 3:

(pog) Ken writes 10 positive integers onto a sheet of paper. Joe then asks the following questions:

- How many of the numbers on your paper are less than 2?
- How many of the numbers on your paper are greater than 2?
- How many of the numbers on your paper are equal to 4?

Ken truthfully answers 3 to every question. What is the sum of the numbers on Ken's paper?

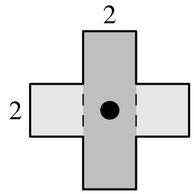
(A) 21 **(B)** 22 **(C)** 23 **(D)** 24 **(E)** 25

Answer (C):

The only positive integer less than 2 is 1, so Ken wrote the numbers $(1, 1, 1)$. The three numbers that Ken wrote that are greater than 2 are $(4, 4, 4)$. Since Ken wrote 10 numbers, we get that $10 - 3 - 3 = 4$ of Ken's numbers are equal to 2. Thus, Ken wrote $(1, 1, 1, 2, 2, 2, 2, 4, 4, 4)$, so the requested sum is $3 + 8 + 12 = \boxed{\textbf{(C) } 23}$. ■

Problem 4:

(pog) Two congruent rectangles each with height 2 are stacked on top of each other. The top sheet is then rotated 90° about its center, resulting in the 12-sided polygon as shown.



If the area of this polygon is 30, what is the area of one of the rectangles?

(A) 12 **(B)** $6\sqrt{6}$ **(C)** 15 **(D)** $5\sqrt{10}$ **(E)** 17

Answer (E):

Let the area of one rectangle be a. Then $2a - 4 = 30$, so $a = \boxed{\textbf{(E) } 17}$. ■

Problem 5:

(DeToasty3) Let a, b, and c be consecutive positive integers, and let p, q, and r also be consecutive positive integers, both not necessarily in order. Given that $a \cdot p = 161$ and $b \cdot q = 189$, what is $c \cdot r$?

(A) 128 **(B)** 144 **(C)** 160 **(D)** 176 **(E)** 192

Answer (D):

The prime factorization of 161 is $23 \cdot 7$. Hence, since a and p are positive integers, $\{a, p\}$ is either $\{23, 7\}$ or $\{161, 1\}$. Since a, b, and c are consecutive positive integers, any two of them differ in absolute value by 2 or less.

If $\{a, p\} = \{161, 1\}$ and $a = 161$ (without loss of generality), then $159 \le b \le 163$. However, since b is an integer and $b \cdot q = 189$, b is a divisor of 189, but no values of b in range are divisors of 189. Hence, $\{a, p\} = \{161, 1\}$ is not possible.

Thus, $\{a, p\} = \{23, 7\}$. Without loss of generality, assume that $a = 23$ and $p = 7$. Then, $21 \le b \le 25$. In this range, only $b = 21$ is a divisor of 189. This implies that $q = 9$. Since $\{a, b, c\}$ and $\{p, q, r\}$ are both sets of three consecutive integers, the only possibility is $c = 22$ and $r = 8$, implying that $c \cdot r = \boxed{\textbf{(D) } 176}$. ■

Problem 6:

(yusufsheikh2207) If x and y are real numbers such that $\sqrt{x\sqrt[3]{y}} = 4$ and $\sqrt{y\sqrt[3]{x}} = 9$, what is xy?

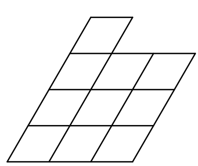
(A) 36 **(B)** 72 **(C)** 108 **(D)** 144 **(E)** 216

Answer (E):

Taking the sixth power of both equations, we get that $x^3 \cdot y = 4^6$ and $y^3 \cdot x = 9^6$. Thus, $x^4y^4 = 4^6 \cdot 9^6$, so $xy = \sqrt[4]{2^{12} \cdot 3^{12}} = 2^3 \cdot 3^3 = \boxed{\textbf{(E) } 216}$. ■

Problem 7:

(pog) How many parallelograms are in the diagram below?



(A) 18 **(B)** 31 **(C)** 36 **(D)** 39 **(E)** 40

Answer (E):

There are two cases to consider: if we do not include the topmost parallelogram, we are simply choosing a parallelogram from a grid of 3×3 parallelograms. There are $\binom{4}{2}$ ways to choose the vertical edges of the parallelogram, and $\binom{4}{2}$ ways to choose the horizontal edges of the parallelogram, for a total of $\binom{4}{2} \cdot \binom{4}{2} = 36$ parallelograms that do not include the topmost parallelogram. Then, there are an additional 4 parallelograms that include the topmost parallelogram, for an answer of $36 + 4 = \boxed{\textbf{(E)}\ 40}$. ■

Problem 8:

(pog) In a survey, each respondent is either a truth-teller or a liar and has a favorite number of either 2 or 5, but not both. Truth-tellers always tell the truth, and liars always lie. Each respondent answered two questions:

- Is your favorite number 2?
- Is your favorite number either 2 or 5?

If 38 liars have a favorite number of 2, 87 respondents answered "Yes" to the first question, and 106 people answered "No" to the second question, how many truth-tellers have a favorite number of 2?

(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

Answer (D):

The second question must be true. If 106 respondents answered "No" to the second question, then there are 106 liars. Hence, $106 - 38 = 68$ liars have a favorite number of 5. Consequently, 68 liars answered "Yes" to the first question, and the other $87 - 68 = \boxed{\textbf{(D)}\ 19}$ people are the truth-tellers that have a favorite number of 2. ■

Problem 9:

(pog) The number of positive divisors of the number $(12!)^{26}$ can be written as

$$\underline{1}\,\underline{3}\,\underline{2}\,\underline{A}\,\underline{0}\,\underline{3}\,\underline{7}\,\underline{6}\,\underline{6}\,\underline{B},$$

where A and B are digits. What is the ordered pair (A, B)?

(A) $(1, 7)$ **(B)** $(2, 6)$ **(C)** $(6, 3)$ **(D)** $(7, 6)$ **(E)** $(9, 8)$

Answer (A):

Note that $(12!)^{26}$ is a perfect square (which always has an odd number of divisors), so B must be odd. This eliminates answer choices **(B)**, **(D)**, and **(E)**.

As well, $12!$ has exactly one factor of 11, so $(12!)^{26}$ will have 11^{26} in its prime factorization. Thus, the number of positive divisors of $(12!)^{26}$ will be divisible by $26 + 1 = 27$. Hence, $1 + 3 + 2 + A + 0 + 3 + 7 + 6 + 6 + B \equiv 0 \pmod 9$, so $A + B \equiv 8 \pmod 9$, eliminating answer choice **(C)**. The only answer choice that works is $\boxed{\textbf{(A)}\ (1, 7)}$. ■

Problem 10:

(pog) A set S consists of 20 consecutive even integers, one of which is equal to 20. Ryan is finding the sum of the 20 elements of S, but he subtracts 20 instead of adding 20, resulting in an incorrect sum that is a multiple of 100. What is the sum of all possible values of the largest element of S?

(A) 108 **(B)** 122 **(C)** 136 **(D)** 150 **(E)** 164

Answer (E):

Let k be the largest element of S. Then, the correct sum of the elements in S is equal to

$$k + (k - 2) + (k - 4) + \cdots + (k - 38) = 20k - 380.$$

Since Ryan subtracted 20 instead of adding 20, his sum is 40 less than the correct sum, so it is equal to $20k - 420$. Hence, $20k - 420 \equiv 0 \pmod{100}$, so $k \equiv 1 \pmod 5$. Since k is even, we get that $k \equiv 6 \pmod{10}$.

Note that 20 must be an element of S, so k is at most $20 + 38 = 58$ and at least 20. The possible values of k are $\{26, 36, 46, 56\}$, for a sum of $\boxed{\textbf{(E)}\ 164}$. ■

Problem 11:

(pog) Ayaka and Judo are each given a whole number greater than 23 and less than 35. Ayaka is also told the units digit of Judo's number, while Judo forgets the units digit of his own number. They are told that both of their numbers are greater than 23 and less than 35. Both of them know where each digit they remember is located in their two numbers, that Ayaka knows the units digit of Judo's number, and that they could have the same number.

- **Ayaka:** I don't know whether my number is less than your number.
- **Judo:** Oh, then I know whose number is larger.
- **Ayaka:** The positive difference between our numbers is 5.

What is the sum of Ayaka and Judo's numbers?

(A) 53 **(B)** 55 **(C)** 59 **(D)** 61 **(E)** 63

Answer (E):

We will go statement by statement.

Statement 1:

If Ayaka is unsure, then the units digit of Judo's number must be 4. If the units digit of Judo's number was not 4, then Ayaka could just determine Judo's number and figure out if her number is greater than Judo's number or not. (If the two numbers are equal, the answer would be no.)

As well, Ayaka's number cannot be 34, because then she would know her number must not be less (i.e. must be greater than or equal to) than Judo's number regardless of if it was 24 or 34, which contradicts her statement.

Statement 2:

Judo now knows that his units digit is 4 and that Ayaka's number is not equal to 34.

Case 1: Judo remembers the tens digit 2.

If Judo remembered the tens digit 2, then he now knows his number is 24. However, this doesn't corroborate with his statement, as he knows Ayaka's number could still be 24, so he could not determine for certain whose number is larger.

Case 2: Judo remembers the tens digit 3.

If Judo remembered the tens digit 3, then he now knows his number is 34. He would then know that his number is larger than Ayaka's as he knows her number is not 34.

Hence, Judo's number must be 34.

Statement 3:

Based on this statement, we find that Ayaka's number is 29, so our answer is $29 + 34 = \boxed{\textbf{(E) } 63}$. ■

Problem 12:

(DeToasty3) A line passes through the point $A(5, 4)$ and has slope -8. A second line passes through the point $B(1, 6)$ and intersects the first line at a point C, equidistant from A and B. What is the slope of the second line?

$\textbf{(A) } \frac{1}{3} \qquad \textbf{(B) } \frac{2}{5} \qquad \textbf{(C) } \frac{1}{2} \qquad \textbf{(D) } \frac{6}{11} \qquad \textbf{(E) } \frac{4}{7}$

Answer (E):

Let ℓ_1 and ℓ_2 be the first two lines.

Since C is equidistant from A and B, it lies on the perpendicular bisector of $\overline{AB}$, which will be defined as line ℓ_3. Line ℓ_1 is given by the equation $y - 4 = -8(x - 5)$. If M is the midpoint of $\overline{AB}$, then $M = (3, 5)$. In addition, ℓ_3 passes through M and is perpendicular to AB. Since line AB has slope $-\frac{1}{2}$, so the equation for ℓ_3 is $y - 5 = 2(x - 3)$.

By definition, ℓ_1 and ℓ_3 intersect at C, where the solution to the system is $(x, y) = (\frac{9}{2}, 8)$. Then, since ℓ_2 contains B and C, its slope is

$$\frac{8-6}{\frac{9}{2}-1} = \boxed{\textbf{(E) } \frac{4}{7}}.$$

■

Problem 13:

(DeToasty3) Given that

$$\log_2 5 \cdot \log_3 6 \cdot \log_4 7 \cdot \log_5 8 = \log_2 a + \log_3 a,$$

what is the nearest integer to a?

$\textbf{(A) } 15 \qquad \textbf{(B) } 17 \qquad \textbf{(C) } 19 \qquad \textbf{(D) } 21 \qquad \textbf{(E) } 23$

Answer (C):

Using the identity $\log_x y = \frac{\log y}{\log x}$, change all the logs to an arbitrary but constant base:

$$\frac{\log 5}{\log 2}\cdot\frac{\log 6}{\log 3}\cdot\frac{\log 7}{\log 4}\cdot\frac{\log 8}{\log 5} = \frac{\log a}{\log 2}+\frac{\log a}{\log 3}$$

$$\implies \frac{\log 6\cdot\log 7\cdot\log 8}{\log 2\cdot\log 3\cdot\log 4} = \log a\cdot\left(\frac{1}{\log 2}+\frac{1}{\log 3}\right).$$

This can further be simplified using the identities $\log x^y = y\log x$ and $\log x + \log y = \log xy$ as follows:

$$\frac{\log 6\cdot\log 7\cdot 3\log 2}{\log 2\cdot\log 3\cdot 2\log 2} = \log a\cdot\left(\frac{\log 2+\log 3}{\log 2\cdot\log 3}\right)$$

$$\implies \frac{3}{2}\cdot\log 6\cdot\log 7 = \log a\cdot\log 6 \implies \log a = \frac{3}{2}\log 7$$

$$\implies a = 7\sqrt{7} = \sqrt{343}.$$

Since $(18.5)^2 < 343 < 19^2$, the closest integer to a is $\boxed{\textbf{(C) } 19}$. ■

Problem 14:

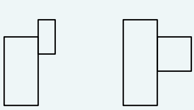
(DeToasty3) A rectangle has perimeter 36. The rectangle is split into three smaller rectangles with dimensions 9-by-4, 6-by-5, and m-by-n. What is $m+n$?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

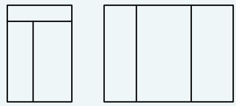
Answer (C):

Consider how three smaller rectangles can be placed to form a larger rectangle. Suppose two rectangles touch and without loss of generality, suppose they touch horizontally.

If there are no horizontal sides that line up as shown in the diagrams below, it is impossible to add another rectangle to the figure to form a bigger rectangle (since there are two spots where the dimensions need to be fixed to form the rectangle).

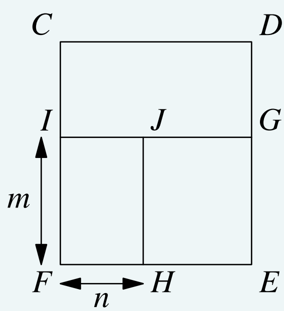


It then follows that the formation of the rectangles will appear like one of the two diagrams below:



The second formation is impossible with the given dimensions of the smaller rectangles, since it would require each rectangle to share a common dimension.

In the first formation, it can be seen that two of the rectangles (label these A and B) share a common dimension. One of these two rectangles must be $m \times n$, as the 9×4 and 6×5 do not share any dimensions. Without loss of generality, let A be the $m \times n$ rectangle, and let m be the shared dimension. See point labels:



Rectangle $CDGI$ is either the 9×4 one or the 6×5. Since the perimeter of $CDEF$ is 36, it follows that $CD + CF = 18$.

If $CDGI$ is the 6×5 rectangle, then $CD + CI = 11$, implying $m = 7$. However, the 9×4 rectangle is forced to be $EGJH$, which does not have a dimension of 7. Thus, $CDGI$ cannot be the 6×5 rectangle.

Hence, $CDGI$ is the 9×4 rectangle, implying $m = 5$. Thus, $EGJH$ is the 6×5 rectangle, implying that $HE = JG = 6$. Since $IG > JG$, it follows that $IG = 9$ and $CI = 4$ (and not the other way around). Thus, $n = 9 - 6 = 3$, implying $m + n = \boxed{\textbf{(C) } 8}$. ■

Problem 15:

(DeToasty3) Let $w = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $z = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$. How many ordered pairs of positive integers (a, b) each at most 6 are there such that $|3w^a + 4z^b| = 5$?

(A) 4 **(B)** 6 **(C)** 8 **(D)** 12 **(E)** 16

Answer (B):

Converting w and z to polar form, $w = \text{cis}(60^\circ)$ and $z = \text{cis}(150^\circ)$. Since these both clearly have magnitude 1, it follows that $|w^a| = 1$ and $|z^b| = 1$. Hence, $3w^a$ and $4z^b$ are some complex numbers with magnitudes 3 and 4, respectively. Let O, W, and P be points in the complex plane such that O is the origin, $W = 3w^a$, and $P = 3w^a + 4z^b$. Then, $|3w^a + 4z^b| = 5$ is equivalent to $OP = 5$. Since $WP = 4$, it follows that $\triangle OWP$ is a 3-4-5 triangle. Thus, PW and OW are perpendicular, implying that the vectors formed by $3w^a$ and $4z^b$ are perpendicular.

Since multiplying a complex number by a real positive constant does not change its argument, the condition is equivalent to w^a and z^b having perpendicular arguments.

By De Moivre's Theorem, $w^a = \text{cis}(60a^\circ)$ and $z^b = \text{cis}(150b^\circ)$. Two complex numbers have perpendicular arguments if and only if they differ by $90^\circ + 180k^\circ$ for some integer k. Thus,

$$60a - 150b \equiv 90 \pmod{180} \implies 2a + b \equiv 3 \pmod{6}.$$

Since $1 \le a, b \le 6$, for each choice of a, there is a unique choice of b that will satisfy the congruence, leading to **(B)** 6 pairs. ■

Problem 16:

(DeToasty3) There are 2022 members in a math tournament, where 999 members are girls, and the rest are boys. The members are split into 674 groups of 3. For every two members in a group, if at least one of them is a girl, they shake hands. Otherwise, they do not. A member is *handy* if they shake hands with both members in their group. Let N be the maximum number of handy members in the math tournament. What is the sum of the digits of N?

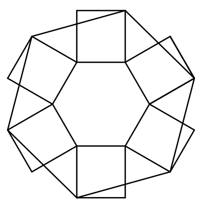
(A) 6 **(B)** 15 **(C)** 16 **(D)** 22 **(E)** 23

Answer (D):

Consider each outcome of a group. If three people are girls, then all three of them are social. If two people are girls and one person is a boy, then all three of them are social. If one person is a girl and two people are boys, then one person is social. If three people are boys, then none of them are social. We want to maximize the number of groups where all three are social by using as few girls as possible, or two girls in a group. Thus, we have 499 groups with two girls and one boy, one group with one girl and two boys, and the other groups with three boys. giving $499 \cdot 3 + 1 = 1498$, so our answer is $1 + 4 + 9 + 8 = \boxed{\textbf{(D)}\ 22}$. ■

Problem 17:

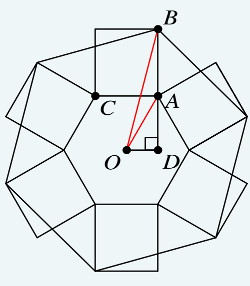
(DeToasty3) In the figure below, six congruent rectangles are glued to each of the sides of a regular hexagon with side length 2, and six of the vertices of the rectangles are connected to form a regular hexagon with side length 4. The length of a side of one of the rectangles not equal to 2 can be written as $\sqrt{m} - \sqrt{n}$, where m and n are positive integers. What is $m + n$?



(A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 19

Answer (D):

See the diagram below for point labels, where O is the center of both hexagons and D is the foot of the altitude from O to line AB.



Let s be the side length of each of the rectangles other than 2. Then, $AB = s$. By a property of regular hexagons, the distance from its center to any of its vertices is the same as the hexagon's side length. Thus, $OA = 2$ and $OB = 4$. Clearly, $\angle OAB = \angle OAC + \angle CAB = 60^\circ + 90^\circ = 150^\circ$. Hence, $\angle OAD = 30^\circ$. It follows that $\triangle AOD$ is a 30-60-90 triangle. It immediately follows that $OD = 1$ and $DA = \sqrt{3}$. Then, by Pythagorean Theorem on

$\triangle ODB$,

$$1^2 + (s+\sqrt{3})^2 + 4^2 \implies s+\sqrt{3} = \pm\sqrt{15} \implies s = \sqrt{15} - \sqrt{3}.$$

Thus, $m + n = \boxed{\textbf{(D) } 18}$.

Problem 18:

(Oxymoronic15) Let $\triangle ABC$ be inscribed in a circle with radius 6. If

$$\sin(\angle ABC) \cdot \sin(\angle BCA) \cdot \sin(\angle CAB) = \frac{17}{24},$$

what is the area of $\triangle ABC$?

(A) 48 **(B)** 51 **(C)** 54 **(D)** 57 **(E)** 60

Answer (B):

By Extended Law of Sines, $\sin(\angle ABC) = \frac{AC}{12}$ and $\sin(\angle BCA) = \frac{AB}{12}$. Thus,

$$AB \cdot AC \cdot \sin(\angle CAB) = \frac{17}{24} \cdot 12^2 = 102.$$

However, by the Law of Sines area formula, the area of $\triangle ABC$ is equal to

$$\frac{1}{2} \cdot AB \cdot AC \cdot \sin(\angle CAB),$$

implying the answer is $\frac{1}{2} \cdot 102 = \boxed{\textbf{(B) } 51}$.

Problem 19:

(HrishiP) Let $P(x)$ be a polynomial with degree 3 and roots r, s, and t with sum 25 such that the coefficient of the x^3 term is 1, and

$$(r+s)(s+t)(t+r) = 2500 \quad \text{and} \quad \left(\frac{1}{r}+\frac{1}{s}\right)\left(\frac{1}{s}+\frac{1}{t}\right)\left(\frac{1}{t}+\frac{1}{r}\right) = 100$$

are satisfied. If the constant term of $P(x)$ is positive, the value of $P(1)$ is equal to $\frac{m}{n}$ for relatively prime positive integers m and n. What is $m + n$?

(A) 406 **(B)** 407 **(C)** 408 **(D)** 409 **(E)** 410

Answer (D):

Let $P(x) = x^3 - 25x^2 + bx + c$. The desire is to find the value of $1 - 25 + b + c$. By Vieta's, $r + s + t = -\frac{-25}{1} = 25$. Then,

$$(25 - t)(25 - r)(25 - s) = 420.$$

We note that this is telling us $P(25) = 420$, or

$$\begin{aligned} 25^3 - 25(25^2) + 25b + c &= 2500 \\ 25b + c &= 2500. \end{aligned}$$

Next, we see that the second equation is equivalent to

$$\begin{aligned} \left(\frac{r+s}{rs}\right)\left(\frac{s+t}{st}\right)\left(\frac{t+r}{tr}\right) &= \frac{(r+s)(s+t)(t+r)}{(rst)^2} \\ &= \frac{2500}{(rst)^2} \\ &= \frac{2500}{c^2}, \end{aligned}$$

where the last equality is from Vieta's. Then, $\frac{2500}{c^2} = 100$ or $c^2 = 25$. If we solve the system

$$25b + c = 2500, \quad c^2 = 25,$$

we get solutions $(b, c) = \left(\frac{501}{5}, -5\right), \left(\frac{499}{5}, 5\right)$. However, $c > 0$, so the last solution is the only one that works. Then,

$$P(1) = 1 - 25 + \frac{499}{5} + 5 = \frac{404}{5},$$

so $m + n = \boxed{\textbf{(D) } 409}$. ■

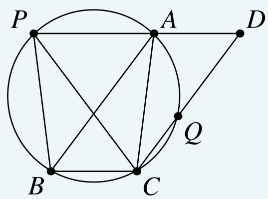
Problem 20:

(DeToasty3) Let parallelogram $ABCD$ have $BC = 5$, $\angle ABC < 90°$, and $\angle ACB > 90°$. Let line AD and side $\overline{CD}$ intersect the circle passing though A, B, and C at $P \neq A$ and $Q \neq C$, respectively. If $CP = 10$ and $CQ = 4$, what is AP?

(A) $\frac{48}{7}$ **(B)** 7 **(C)** $\frac{36}{5}$ **(D)** $\frac{15}{2}$ **(E)** 8

Answer (B):

Let ω be the circumcircle of $\triangle ABC$. Since it is given that $\angle ACB$ is obtuse, the center of ω would be outside $\triangle ABC$. By constructing parallelogram $ABCD$ from the locations of A, B, and C, it is clear that D will be outside ω and Q will be between C and D. Thus, A will be between P and D.



Since $ACBP$ is a quadrilateral inscribed in a circle with $\overline{AP} \parallel \overline{BC}$, $ACBP$ is an isosceles trapezoid. Thus, $AB = CP = 10$. Then, by the parallelogram, $DC = AB = 10$ and $AD = BC = 5$. With $CQ = 4$, it follows that $QD = 6$. Then, by Power of a Point at D,

$$DA \cdot DP = DQ \cdot DC \implies 5 \cdot DP = 6 \cdot 10 \implies DP = 12.$$

Hence, $AP = \boxed{\textbf{(B) } 7}$. ■

Problem 21:

(john0512) For each positive integer n, let $f_1(n) = n!$, and for $k \geq 2$, let

$$f_k(n) = f_{k-1}(1) \cdot f_{k-1}(2) \cdot \ldots \cdot f_{k-1}(n).$$

Let N be the largest integer such that $f_4(10)$ is divisible by 2^N. What is the sum of the digits of N?

(A) 4 **(B)** 15 **(C)** 16 **(D)** 20 **(E)** 21

Answer (E):

By taking $n-1$ instead of n in the function,

$$f_k(n-1) = f_{k-1}(1) \cdot f_{k-1}(2) \cdot \ldots \cdot f_{k-1}(n-1),$$

so the given relation can be rewritten as

$$f_k(n) = f_k(n-1) \cdot f_{k-1}(n). \quad (*)$$

Let $\nu_2(x)$ denote the largest integer a such that 2^a divides x. Then, $\nu_2(f_4(10)) = N$ and by $(*)$,

$$\nu_2(f_k(n)) = \nu_2(f_k(n-1)) + \nu_2(f_{k-1}(n)). \quad (\star)$$

By the definition of a factorial, $f_1(n) = f_1(n-1) \cdot n$, implying that $\nu_2(f_1(n)) = \nu_2(f_1(n-1)) + \nu_2(n)$. Using the base case of $\nu_2(f_k(1)) = 0$ for all $k \geq 1$ and $(\star)$, a recursion table can be filled out for $\nu_2(f_k(n))$:

k \ n	1	2	3	4	5	6	7	8	9	10
1	0	1	1	3	3	4	4	7	7	8
2	0	1	2	5	8	12	16	23	30	38
3	0	1	3	8	16	28	44	67	97	135
4	0	1	4	12	28	56	100	167	264	399

Thus, $N = 399$, so our answer is $\boxed{\textbf{(E) } 21}$. ■

Problem 22:

(treemath) Freida writes down the number 1 on a blackboard. Then, she repeatedly picks a written number n at random, and she writes down either $2n$, $3n$ or $4n$, where each number is equally likely to be written. On average, what is the sum of the first 20 numbers she writes down?

(A) 1330 **(B)** 1540 **(C)** 1600 **(D)** 1760 **(E)** 1960

Answer (B):

Define E_m as the expected value of the mth number written. We claim that $E_m = \frac{m(m+1)}{2}$ for $m \geq 1$, and we will prove this with strong induction. The claim holds for the base case of $m = 1$.

Assume the induction hypothesis holds for each of $m = 1, 2, 3, \ldots, k$ for some integer k. It now suffices to show this implies it will hold for $m = k + 1$.

Since Freida determines the next number by randomly choosing one of the k numbers currently on the board and then multiplying it by either 2, 3, or 4. By linearity of expectation,

$$E_{k+1} = \frac{2+3+4}{3} \cdot \frac{E_1 + E_2 + \cdots + E_k}{k} = \frac{3}{2k} \cdot \sum_{a=1}^{k} (a^2 + a).$$

Using the well-known sum of integers and sum of squares formulae,

$$E_{k+1} = \frac{3}{2k}\left(\frac{k(k+1)(2k+1)}{6} + \frac{k(k+1)}{2}\right) = \frac{3}{2} \cdot \frac{(k+1)(2k+4)}{6}$$

$$= \frac{(k+1)(k+2)}{2},$$

which completes the induction. ■

Thus, the requested sum is $\sum_{k=1}^{20} E_k = \frac{1}{2}\sum_{k=1}^{20}(k^2+k) = \boxed{\textbf{(B) } 1540}$. ■

Problem 23:

(stayhomedomath) In acute $\triangle ABC$ with $AC < BC$, the perpendicular bisector of $\overline{AB}$ meets lines AB, BC, and AC at D, E, and F, respectively. If $AD = 5$, $BE = 13$, and the area of $\triangle ABC$ is 14 units greater than that of $\triangle ADF$, what is AF^2?

(A) 650 **(B)** 701 **(C)** 754 **(D)** 809 **(E)** 866

Answer (D):

Henceforth, let the area of a polygon $\mathcal{P}$ be denoted by $[\mathcal{P}]$. We shall first explain the setup of the configuration. Clearly, D is the midpoint of $\overline{AB}$.

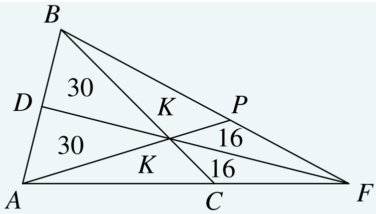
Let ℓ_1, ℓ_2, and ℓ_3 be the lines perpendicular to $\overline{AB}$ passing through A, D, and B, respectively. Then, in order for $\triangle ABC$ to be acute, C must lie between ℓ_1 and ℓ_3; and since $AC < BC$, C lies between ℓ_1 and ℓ_2. As ℓ_2 is the perpendicular bisector of $\overline{AB}$, it follows that E lies in between D and F, with E on $\overline{BC}$ and F on the extension of $\overline{AC}$ past C.

Since $AD = BD = 5$, $DE = 12$ by the Pythagorean Theorem, and $[BDE] = \frac{1}{2} \cdot 5 \cdot 12 = 30$. Consequently,

$$[ABC]-[ADF] = ([BDE]+[ACED])-([CEF]+[ACED]) = 30-[CEF],$$

and setting this equal to 14 gives $[CEF] = 16$.

Now, extend $\overline{AE}$ past E to meet $\overline{BF}$ at P. By symmetry, $[ADE] = [BDE] = 30$ and $[CEF] = [PEF] = 16$; accordingly denote $[ACE] = [BPE] = K$.



By area ratios,

$$\frac{[AEF]}{[PEF]} = \frac{AE}{PE} = \frac{[ABE]}{[PBE]} \implies \frac{K+16}{16} = \frac{60}{K}.$$

Cross-multiplying, we have that

$$K^2 + 16K - 960 = (K-24)(K+40) = 0 \implies K = 24.$$

Hence, $[ADF] = 30 + 24 + 16 = 70$, and $DF = \frac{2[ADF]}{AD} = 28$. Therefore, $AF^2 = 5^2 + 28^2 = \boxed{\textbf{(D) } 809}$. ■

Problem 24:

(bissue) Define a sequence of real numbers $a_1, a_2, \ldots$ by $a_1 = 2$, $a_2 = 9$, and $a_{n+2} = 2a_{n+1} + a_n$ for all positive integers n. If

$$\sum_{k=1}^{\infty} \frac{a_k a_{k+1}}{(a_{k+1}^2 - a_k^2)^2} = \frac{1}{m},$$

where m is a positive integer, what is the sum of the digits of m?

(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

Answer (A):

Manipulate the expression being summed as follows:

$$\frac{a_k a_{k+1}}{(a_{k+1}^2 - a_k^2)^2} = \frac{a_k a_{k+1}}{(a_{k+1}+a_k)^2(a_{k+1}-a_k)^2} = \frac{1}{4} \cdot \frac{4a_k a_{k+1}}{(a_{k+1}+a_k)^2(a_{k+1}-a_k)^2}$$

$$\frac{1}{4} \cdot \frac{(a_{k+1}+a_k)^2 - (a_{k+1}-a_k)^2}{(a_{k+1}+a_k)^2(a_{k+1}-a_k)^2} = \frac{1}{4} \cdot \left(\frac{1}{(a_{k+1}-a_k)^2} - \frac{1}{(a_{k+1}+a_k)^2} \right).$$

By the given recursion, $a_{k+2} - a_{k+1} = a_{k+1} + a_k$. Hence, the expression above is

$$\frac{1}{4} \cdot \left(\frac{1}{(a_{k+1}-a_k)^2} - \frac{1}{(a_{k+2}-a_{k+1})^2} \right).$$

It now becomes clear when this expression is summed from $k = 1$ to ∞, it will telescope, leaving behind $\frac{1}{4} \cdot \frac{1}{(a_2 - a_1)^2} = \frac{1}{196}$.

Thus, $m = 196$, and the sum of its digits is $\boxed{\textbf{(A) } 16}$. ■

Problem 25:

(DeToasty3) Convex quadrilateral $ABCD$ has acute angles $\angle A$ and $\angle D$, obtuse angles $\angle B$ and $\angle C$, $AB = BC = CD = 10$, and $AC + BD = 24\sqrt{2}$. Let M and N be the midpoints of $\overline{AC}$ and $\overline{BD}$, respectively. If $MN = 5$, what is AD^2?

(A) 288 **(B)** 320 **(C)** 384 **(D)** 432 **(E)** 486

Answer (C):

Clearly, with $AB = BC$ and $BC = CD$, it follows that $BM \perp AC$ and $CN \perp BD$. Also, BM bisects $\angle B$ and CN bisects $\angle C$. Let lines BM and CN intersect at E.

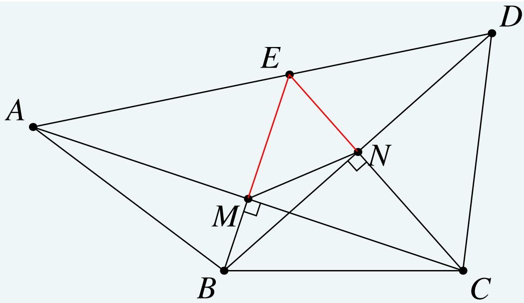
Clearly, $\angle EBC$ and $\angle ECB$ must be acute or $\angle ACB$ or $\angle BCD$ will be greater than 180°. Since $\angle B$ and $\angle C$ are obtuse, they are greater than 90°, so $\angle EBC + \angle BCE > 90^\circ$. Thus, $\angle BEC$ is acute, so $\triangle BCE$ is an acute triangle. As a result, M lies between B and E, and N lies between C and E.

Clearly, quadrilateral $BCNM$ is cyclic, implying that $\triangle ENM \sim \triangle EBC$. Furthermore, $\frac{NM}{BC} = \frac{EN}{EB} = \cos(\angle BEC)$ by right triangle $\triangle ENB$. Since $\frac{NM}{BC} = \frac{1}{2}$, it follows that $\angle BEC = 60^\circ$.

Since EM and AC are perpendicular with M being the midpoint of $\overline{AC}$, $\triangle EMC \cong \triangle EMA$. Thus, $\angle MEA = 60^\circ$. Similarly, $\triangle ENB \cong \triangle END$, implying $\angle NED = 60^\circ$. Since

$$\angle AEM + \angle BEC + \angle NED = 180^\circ,$$

A, E, and D are collinear.



Furthermore, $\triangle AEC$ and $\triangle BED$ are 30-120-30 isosceles triangles. Hence,

$$AD = AE + ED = \frac{1}{\sqrt{3}}(AC + BD) = 8\sqrt{6}.$$

Thus, $AF^2 = \boxed{\textbf{(C)}\ 384}$. ■