1. Note that the height from C to $\overline{AB}$ has length 8, forming two special right triangles. Suppose that $AD = x$. Then by similar triangles, we have $(6-x)\cdot\frac{5}{3} = x$, and solving gives $x = \frac{15}{4}$. Our answer is **019**.

2. The three sets must be of the form:

$$\{?,?,?,6\} \quad \{?,?,?,9\} \quad \{?,?,?,12\}$$

Note that five positive integers are less than 6. We can choose three to be in the set with six. Now out of the two remaining numbers, along with 7 and 8, we must choose three to be in a set with nine. There are three remaining numbers, and these determine the set with twelve. Thus, the number of partitions is $\binom{5}{3}\binom{4}{3} =$ **040**.

3. The proportions of books remaining are $1, 1-\frac{m}{n}$, and $1-\frac{m}{n}-\frac{1}{m}(1-\frac{m}{n})$. The differences must be equal, so it follows that

$$1-\left(1-\frac{m}{n}\right) = 1-\frac{m}{n}-\left(1-\frac{m}{n}-\frac{1}{m}\left(1-\frac{m}{n}\right)\right) \implies \frac{m}{n} = \frac{1}{m}-\frac{1}{n}$$

Multiplying both sides by mn, we have $n = m^2+m$, and $\frac{m}{n} = \frac{1}{m+1}$. We thus need the sum of all m where $m+1$ divides 240, which can be computed to be **724**.

4. Note that the graph drawn does not pass through any lattice point, since when x is an integer, y is guaranteed to be irrational. Thus, we simply count the number of intersections of $y = x^2-\sqrt{2}$ with horizontal and vertical lines. We further simplify our count by noting symmetry across the y-axis and multiplying the count for $x > 0$ by 2. For $x > 0$, notice that $x^2-\sqrt{2}$ is increasing. It starts at $-2 < y = -\sqrt{2} < -1$, so it crosses the horizontal lines $y = -1$, $y = 0$, $y = 1$, ... $y = 99$ until it exits the paper. This gives 101 intersections. At the point it exits the paper, $x^2-\sqrt{2} = 100$, so $x^2 = 100+\sqrt{2}$ gives us that $10 < x < 11$. Therefore, lines $x = 1$, $x = 2, \ldots x = 10$ are crossed. This gives 10 intersections. For each of the 111 intersections, we go from one unit square to another. Thus, we have 112 unit squares due to these 111 dividers. The answer is $112\cdot 2 =$ **224** due to symmetry across the y-axis.

5. We choose three vertices that are spaced the furthest apart. Since the area is fixed, this creates the largest possible triangle inside the regular hexagon. Suppose the regular hexagon has side length s. The largest possible triangle occurs when we choose two vertices spaced $s\sqrt{3}$ apart on the bottom face and one vertex on the top face that is $2s$ units away from both of those vertices. Then our desired triangle is isosceles with lengths $2s, 2s, s\sqrt{3}$, and area

$$\frac{s\sqrt{3}}{2}\cdot\sqrt{(2s)^2-\left(\frac{s\sqrt{3}}{2}\right)^2} = \sqrt{39} \implies s = 2$$

The volume of the hexagon is thus $\frac{3\sqrt{3}\cdot 2^2}{2}\cdot 2 = 12\sqrt{3}$, and $(12\sqrt{3})^2 =$ **432**.

6. The idea is to think in base 4^i-3^i, so the places are $4^0-3^0 = 0$, $4^1-3^1 = 1$, and so on $(0, 1, 7, 37, 175, 781)$. Since the digits are just the same as in base 3, we see that there is no interference between the places, except 0 and 1, which is easily resolved by ignoring 0. Thus, the 5 nonzero places can be multiplied by one of $0, 1$, and 2 and then added together to be combined into $m-n$, giving 243 ways without bound (00000 is achievable due to the discarded 0 place being a filler, say $m = n = 1$.) The maximum is $2(1+7+37+175+781) = 2002$, so we do need to remove 2001 and 2002 from our count, giving **241** possible differences.

7. For symmetry we work with $\frac{a+b+c}{a}$ and $\frac{a+b+c}{b}$. Let p and q represent these two expressions. Note that
$$\frac{a+b+c}{c} = \frac{1}{1-\frac{1}{p}-\frac{1}{q}} = \frac{1}{1-\frac{p+q}{pq}}.$$
Now note that by Vieta's Formulas, $(p-1)+(q-1) = 574$ and $(p-1)(q-1) = 17$, implying $p+q = 576$ and $pq = 592$. Then
$$\frac{1}{1-\frac{p+q}{pq}} = \frac{1}{\frac{16}{592}} = \frac{592}{16} = 37.$$
Since we are looking for 1 less than this quantity, our answer is **036**.

8. The problem asks for the unique denominator n such that for any fraction $\frac{m}{n}$ where m is a positive integer, 0.08 is the smallest number that cannot be approximated by $\frac{m}{n}$. It is easy to see that for any $n \geq 100$, 0.08 can be approximated to $\frac{m}{n}$ for some positive integer $m < n$. Thus, we shift our goal to n being as close to 100 as possible, with $\frac{m}{n}$ being 0.005 or more above 0.08 and $\frac{m-1}{n}$ being 0.005 or more below 0.08. Writing this algebraically, we have
$$\frac{m-1}{n} + \frac{1}{200} < \frac{2}{25} < \frac{m}{n} - \frac{1}{200}$$
Letting $m = 8$ generates the value of n closest to 100, if such n exists. So
$$\frac{7}{n} < \frac{2}{25} - \frac{1}{200} \quad \text{and} \quad \frac{2}{25} + \frac{1}{200} < \frac{8}{n}$$
$$\implies \frac{7}{n} < \frac{3}{40} \qquad \frac{17}{200} < \frac{8}{n}$$
We must have $3n > 280$ and $17n < 1600$. Now we multiply the first inequality and 17 and the second by 3 to obtain $4760 < 51n < 4800 \implies 93.\overline{3} < n < 94.11\ldots$ Therefore, $n =$ **094**.

9. Points are defined as shown.

Since $PQDC$ is cyclic, we have $\angle PQC = \angle PDC = 90^\circ$. Let $\angle ICD = a$. Then we have $\angle QHI = \angle HIE = \angle ECD = 2a$. But since $\angle QIP = \angle DIC = 90 - a$, we have $QH = HI$. Then since triangle QPI is right, H is the midpoint of PI. To finish, note that since $\angle BHD = \angle AIE = 2a$ and $\angle HDB = \angle IEA = 90^\circ$, we have $\triangle BHD \sim \triangle AIE$. Thus $\frac{BD}{AE} = \frac{HD}{IE} = \frac{3}{2}$. Since $BD = DC = CE$, we know $AC = \frac{5}{3}DC$. Now $ID = IE = 6$, so $AI = \frac{5}{3}IE = 10$. Then $AD = 16$, so $DC = 12$ and we have $[ABC] = AD \cdot DC =$ **192**.

10. Let a, b, c denote the lengths of BC, CA, and AB, respectively, let $s = \frac{a+b+c}{2}$, and let r and r_A be the inradius and the A-exradius, respectively. Drawing the incircle with center I, tangent to sides BC, CA, and AB at D, E, and F, respectively, we see that $BF = BD$ and $CE = CD$. Also, note that $AEIF$ is a square, so $AE = AF = s - a = r$. The excircle configuration happens to be similar. Let the excircle with center I_A be tangent to BC, CA, and AB at D', E', and F', respectively. Then, $BF' = BD'$ and $CE' = CD'$. Once again, $AE'I_AF'$ is a square, with side length $AF' = AB + BF' = c + (s-c) = s = r_A$. We observe that $r_A - r = s - (s-a) = a$, so in fact we are adding up the distinct possible values of $a + r$

(actually r is fixed, so the distinctness depends on a). Notice that $a = BC = BD + CD = BF + CE = (AB - AF) + (AC - AE) = b + c - 2r$. However, the Pythagorean theorem also yields $a^2 = b^2 + c^2$, so

$$(b + c - 2r)^2 = b^2 + c^2.$$

Expanding and simplifying, we get

$$2r^2 - 2rb - 2rc + bc = 0.$$

Using Simon's Favorite Factoring Trick, we get

$$(b - 2r)(c - 2r) = 2r^2.$$

Thus, the solutions must satisfy $b - 2r = d$ and $c - 2r = \frac{2r^2}{d}$ where d is a positive integer divisor of $2r^2$. Recall that we want distinct solutions for $a + r = (b + c - 2r) + r = (b - 2r) + (c - 2r) + 3r$ (also note that $r = 10$ is fixed). Therefore, the sum

$$\sum_{d|2r^2} d + \frac{2r^2}{d} + 3r$$

overcounts by a factor of 2, so the correct sum is

$$\sum_{d|2r^2, d \le \sqrt{2r^2}} d + \frac{2r^2}{d} + 3r = \sigma(2r^2) + 3r \cdot \frac{\tau(2r^2)}{2},$$

(σ and τ denote sum of divisors and number of divisors, respectively) which we finally (and safely) evaluate to $\sigma(200) + 30 \cdot \frac{\tau(200)}{2} = 31 \cdot 15 + 30 \cdot 6 =$ **645**.

11. The idea is to approach with the principle of inclusion and exclusion on the first two conditions (which are of course, symmetric). Generalize 720 to $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ (One can probably deduce that only the exponents matter). Before we start counting, note that we are limited to the k primes when constructing a and b. We first count $ab | n^2$ with no restrictions. Note that the count is multiplicative across the prime powers, so we only demonstrate p^e in general. We need non-negatives $v_p(a) = e_a$ and $v_p(b) = e_b$ to satisfy $e_a + e_b \le 2e$. If one introduces the pseudopower e_c to complete the sum to $e_a + e_b + e_c = 2e$, by stars and bars there are $\binom{2e+2}{2}$ possibilities. Thus no restrictions will give us

$$\prod_{i=1}^{k} \binom{2e_i + 2}{2}.$$

We then subtract away those satisfying $a | n$ and $ab | n^2$ (and the same thing for $b | n$). See $n^2 = p_1^{2e_1} p_2^{2e_2} \dots p_k^{2e_k}$ and $a = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ where $0 \le a_i \le e_i$ for all i. For each case of a, we see that $b_i = v_{p_i}(b)$ is in the closed interval $[0, 2e_i - a_i]$. This means that over the $\prod_{i=1}^{k}(e_i + 1)$ cases for a, we are summing a distributive product of k terms. In classic distributive fashion, we collect the factor

$$(2e_i + 1) + (2e_i) + (2e_i - 1) + \dots + (e_i + 1)$$

that we take the product of over all i. Just like that, the number we must subtract off amounts to

$$\prod_{i=1}^{k} \frac{(e_i + 1)(3e_i + 2)}{2}.$$

Note that this is subtracted twice due to the symmetry on a and b. Finally, as promised, we have to add back $a|n, b|n, ab|n^2$, where we can see that the third condition has no meaning. This simply yields

$$\prod_{i=1}^{k}(e_i+1)^2.$$

Our final expression is

$$\prod_{i=1}^{k}\binom{2e_i+2}{2} - 2\prod_{i=1}^{k}\frac{(e_i+1)(3e_i+2)}{2} + \prod_{i=1}^{k}(e_i+1)^2.$$

For $n = 720$, it evaluates to

$$6\cdot 15\cdot 45 - 2(5\cdot 12\cdot 35) + (2\cdot 3\cdot 5)^2 = \mathbf{750}.$$

12. The answers are $n = 32$ and $n = 36$, which both work. Write $n = p_1^{e_1}\cdot p_2^{e_2}\cdot \ldots p_k^{e_k}$ for positive integer exponents e_i. Note that $\tau(n) = (e_1+1)(e_2+1)\ldots(e_k+1)$.We claim that $\frac{\phi(n)}{n} = \frac{p_1-1}{p_1}\cdot\frac{p_2-1}{p_2}\cdot\ldots\frac{p_k-1}{p_k}$. This is well known, but here is a short proof: From the sample containing all the positive integers at most n, the probability that a randomly chosen one is relatively prime to n is $\frac{\phi(n)}{n}$, but it is also the product of independent probabilities that the chosen one is not divisible by $p_1, p_2, \ldots, p_k$. But this is just $\frac{p_1-1}{p_1}\cdot\frac{p_2-1}{p_2}\cdot\ldots\frac{p_k-1}{p_k}$. Proceeding with the problem, we see that

$$(e_1+1)(e_2+1)\ldots(e_k+1) = \tau(n) = \frac{3n}{\phi(n)} = 3\cdot\frac{p_1}{p_1-1}\cdot\frac{p_2}{p_2-1}\cdot\ldots\frac{p_k}{p_k-1}$$

When $k = 1$, the only fraction we can multiply to 3 that keeps the right hand side an integer is $\frac{2}{1}$. Indeed, if $(e_1+1) = 3\cdot\frac{2}{1} = 6$, we receive $p_1 = 2$ and $e_1 = 5$, yielding the solution $2^5 = 32$ When $k = 2$, note that we must have 2 as one of our primes because any other ratio has more powers of 2 on the bottom than on top. We have $3\cdot\frac{2}{1}\cdot\frac{p_2}{p_2-1}$. This has to be an integer, and it must be greater than 6. However, it's maximum value is attained at $p_2 = 3$, where the product $(e_1+1)(e_2+1) = 9$, producing solutions $e_1 = e_2 = 2$ while $p_1 = 2$, $p_2 = 3$, in other words, 36. As for 7 and 8, note that 7 is achievable at $p_2 = 7$, but it's useless because 7 cannot be written as the product of two factors each greater than 1. 8 requires the ratio $\frac{4}{3}$, where $p_2 = 4$ is not a prime. When $k \geq 3$, there are guaranteed more powers of 2 in the denominator of the right hand side than the numerator since every fraction $\frac{p}{p-1}$ other than $\frac{2}{1}$ has an even numerator, so we are done. The answer is $32 + 36 = \mathbf{068}$.

13. Note that there are primary chokepoints at $(4x, 4y)$ for integers x and y. The idea is that the situation is enhancing up-right paths from $(0,0)$ to $(17,17)$ by scaling them up by a factor of 4. If we draw it out, we see that around a certain chokepoint $(4c, 4d)$, the freedom we have comes from turning at the chokepoint. If we turn, we have 5 options multiplied in. If we pass straight through, we don't have any options. Thus, we desire

$$\sum_{i=1}^{33}(\text{Number of } 17\times 17 \text{ up-right paths with } i \text{ turns})\cdot 5^i.$$

An up-right path with exactly i bends can be started with either up or right, say right at first. Then, there is to be $\lceil\frac{i+1}{2}\rceil$ pieces of right and $\lfloor\frac{i+1}{2}\rfloor$ pieces of up, where each piece is of length at least 1 (or 4 in the original problem). By Stars and Bars, we have $\binom{16}{\lceil \frac{i-1}{2} \rceil}$ ways to manage the right pieces, and $\binom{16}{\lfloor \frac{i-1}{2} \rfloor}$ ways to manage the up pieces. This yields $2\binom{16}{\lceil \frac{i-1}{2} \rceil}\binom{16}{\lfloor \frac{i-1}{2} \rfloor}$ up-right paths with exactly i turns. Therefore, our desired sum is

$$2\left(5^1\binom{16}{0}\binom{16}{0}+5^2\binom{16}{0}\binom{16}{1}+5^3\binom{16}{1}\binom{16}{1}+5^4\binom{16}{1}\binom{16}{2}+\cdots+5^{33}\binom{16}{16}\binom{16}{16}\right).$$

Taking this mod 125 gives $2(5+400)=810\equiv 60 \pmod{125}$ and mod 8 gives $2(5^1+5^{33})\equiv 2(2\cdot 5)\equiv 4 \pmod 8$, so our answer is **060**.

14. Observe that $2017\cdot 2081=(2^{11}+2^5+1)(2^{11}-2^5+1)$; for the ease of factoring, we multiply this by 4 to factor it as $(2^{12}+2^6+2)(2^{12}-2^6+2)$. Letting $x=2^6$, we find that $(2^{12}+2^6+2)(2^{12}-2^6+2)$ simplifies to x^4+3x^2+4. Using the x substitution on our first expression, and discarding the 2^{14} (which we'll add back at the end), we have

$$2^{96}-2^{72}+2^{28}=x^{16}-x^{12}+16x^4.$$

Note that

$$(x^4+3x^2+4)(x^4-3x^2+4)=x^8-x^4+16$$

Multiplying this by x^8, the expression changes to $x^{16}-x^{12}+16x^8$. To get rid of the x^8 term, we multiply x^8-x^4+16 by -16, yielding $-16x^8+16x^4-16^2$. Adding this back to $x^{16}-x^{12}+16x^8$ gives us $x^{16}-x^{12}+16x^4-16^2$. All terms with a nonzero degree of x have matching coefficients, and we have $x^{16}-x^{12}+16x^4\equiv 16^2 \pmod{2017\cdot 2081}$. The answer is thus $16^2+2^{14}\equiv \mathbf{640} \pmod{1000}$.

15. Note that since $\frac{AP}{AB}=\frac{AQ}{AC}=2\cos A$, $\triangle APQ\sim\triangle ABC$. Let the midpoint of BC be M. Since AM is the symmedian of $\triangle APQ$, it has length $2\cos A\cdot\frac{AB\cdot AC}{AM}$, where $\frac{AB\cdot AC}{AM}$ is the length of the symmedian of $\triangle ABC$. Thus,

$$AM=2\cos A\cdot\frac{AB\cdot AC}{AM}=2\cos A\cdot\frac{9\cdot 15}{AM},$$

implying that $AM^2=270\cos A$. By the Law of Cosines,

$$BC^2=AB^2+AC^2-2(AB\cdot AC)\cos A=AB^2+AC^2-AM^2,$$

and by Stewart's Theorem,

$$BC^2=2AB^2+2AC^2-4AM^2.$$

This gives us

$$\begin{aligned}AB^2+AC^2-AM^2&=2AB^2+2AC^2-4AM^2\\3AM^2&=AB^2+AC^2\\AM^2&=\frac{AB^2+AC^2}{3}.\end{aligned}$$

Substituting this back into either equation gives us

$$BC^2=\frac{2}{3}(AB^2+AC^2)=\frac{2}{3}(9^2+15^2)=\mathbf{204}.$$