1. Note that $9! = \frac{1}{10} \cdot 10!$, so $1 - \frac{1}{10} = \frac{9}{10} = 90\%$ **(E)** of the people remain in the city.

2. There is only one junior per trio, which is 20% of juniors, and there are thus 5 juniors represented for one junior in a trio (including this one junior). Similarly, two seniors per trio represents 25% of all seniors, so there are 8 total seniors represented for the two seniors in a trio. The ratio is $5 : 8$ **(D)**.

3. Let x and 9 be the numbers erased. The sum of the remaining numbers is $(1+2+\cdots+10) - x - 9 = 55 - (x - 9)$ and we want $\frac{55-(x+9)}{x+9} = \frac{55}{x+9} - 1$ to be an integer (which would make $\frac{55}{x+9}$ an integer). Out of all possibilities, $x = 2$ **(A)** is the only integer in the interval $[1, 10]$ such that $x + 9$ divides 55.

4. Suppose that the square has side length 3. Then the three congruent rectangles used to make the square must share sides with lengths 3 and $\frac{3}{3} = 1$. To form the rectangle that is not a square, we attach each of the three squares to one of the other squares along the side with length 1, forming a 1×9 rectangle. The ratio is $\frac{9}{1} = 9$ **(E)**.

5. Choices (A),(B), and (E) can be disproven by letting $a = 1$. Choice (C) is false: decreasing a, despite b's relation to c, always decreases $a^{\frac{b}{c}}$. Choice **(D)** is true: increasing c decreases $\frac{b}{c}$ and thus increases $a^{\frac{b}{c}}$ where $a < 1$.

6. If we change the operation in front of the number n, we will multiply it by n^2. Therefore we want to maximize n^2 which comes when we pick larger fractions. Here, $\frac{1}{3}$ and $\frac{1}{2}$ are the largest numbers while $\frac{1}{4}$ and $\frac{1}{5}$ are the smallest numbers. The maximum possible result upon changing two operations is $1 \cdot \frac{1}{3} \div \frac{1}{5} \cdot \frac{1}{2} \div \frac{1}{4} = \frac{10}{3}$ **(D)**.

7. The longest diagonal of a regular hexagon are the three diameters of the hexagon's circumcircle that span across the hexagon's vertices. Let s be the length of the hexagon. Then, by dropping the altitudes from two adjacent vertices to the diagonal, we find that the diagonal has length $\frac{s}{2} + \frac{s}{2} + s = 2s$. The equilateral triangle with $2s$ as a length has area $4\sqrt{3}$. By the equilateral triangle area formula, $\frac{(2s)^2\sqrt{3}}{4} = \frac{4s^2\sqrt{3}}{4} = 4\sqrt{3} \implies s = 2$. Thus the perimeter is $6s = 6 \cdot 2 = 12$ **(C)**.

8. The smallest sum of digits is achieved through letting $\heartsuit = 1$ and $\diamondsuit = 0$ for a sum of $3 \cdot 1 + 0 = 3$. The largest sum of digits is achieved through $\heartsuit = 9$ and $\diamondsuit = 8$, yielding $9 \cdot 3 + 8 = 37$. Observe that by parity, the sum of digits is always odd, and any odd number between 3 and 37 can be the sum of the digits. Thus there are $\frac{37-3}{2} = 17$ **(B)** possible sums of digits.

9. The first, second, and third conditions can be achieved by labeling the vertices in the order $(1, 2, 3, 4)$. The fourth condition implies that $1, 2$, and 4 are never adjacent; this is impossible as 2 must always be adjacent to either 1 or 4. Thus 3 **(D)** conditions have possible labelings.

10. By direct observation, we see that $\lfloor \frac{n^2}{2} \rfloor = \lceil \frac{n}{2} \rceil^2$ occurs when $n = 0, 1$, for nonnegative n, and $\lfloor \frac{n^2}{2} \rfloor < \lceil \frac{n}{2} \rceil^2$ for $n > 2$. Additionally, for $n > 2$, $\lfloor \frac{n^2}{2} \rfloor$ increases faster than $\lceil \frac{n}{2} \rceil^2$. For negative n, we find that $n = -3$ is a solution (yielding $4 = 4$.) For $n < -3$, $\lfloor \frac{n^2}{2} \rfloor$ also increases faster than $\lceil \frac{n}{2} \rceil^2$. Verifying that $n = -1, -2$ do not work, we find 3 **(C)** solutions.

11. Notice that the n^{th} section of the list, consisting of strictly decreasing numbers with the maximum number being the first term, has the numbers in the range $[\frac{n(n-1)}{2}+1, \frac{n(n+1)}{2}]$ backwards. Since 2021 is 4 more than $\frac{64 \cdot 63}{2}+1=2017$, its position is $\frac{64 \cdot 65}{2}-4=2076$ **(D)**.

12. Each article of clothing can either be included or not included in the outfit, so there are a total of $(3+1)(4+1)(5+1)=120$ **(D)** possible outfits.

13. Note that in each trapezoid $Q_nP_nP_{n+1}Q_{n+1}$ and trapezoid ACP_0Q_0, the shaded area is twice the unshaded area since the shaded area and unshaded area share the same height but the shaded area's base is twice the length of the that of the unshaded area, so the total unshaded area is $96 \cdot \frac{1}{3}=32$ **(C)**.

14. We perform casework based on the sum of the opposite faces. The minimum possible sum is $1+6=7$ and the maximum possible sum is $6+11=17$. The number of corresponding face pair possibilities for the different sums are manually computed to be $3,3,4,4,5,5,5,4,4,3,3$ being symmetrical about where the sum equals 12. In total, there are

$$2\left(2 \cdot \binom{3}{3}+2 \cdot \binom{4}{3}+\binom{5}{3}\right)+\binom{5}{3}=50 \text{ possibilities } \textbf{(D)}$$

15. Because EG is parallel to AF, $\triangle EBG \sim \triangle ABC$. Let $AE=x$ be the length of the rhombus. Then $EB=14-x$ and $\frac{EB}{EG}=\frac{AB}{BC} \implies \frac{14-x}{x}=\frac{14}{35} \implies x=10$ **(D)**.

16. Note that the digits must be some permutation of the remainders $\{1,1,0\}$ or $\{2,2,0\}$ when divided by three: 1 and 2 cannot be two of the three remainders and all three remainders cannot be uniform. If the remainders are $\{1,1,0\}$ and the zero is not in the front, there are $2 \cdot 3 \cdot 3 \cdot 4=72$ possiblities. If zero is in the front, we discard the zero digit and there are $3 \cdot 3 \cdot 3=27$ possibilities. So there are $72+27=99$ possibilities here. Likewise, the case for $\{2,2,0\}$ also has 99 possiblities. In total, there are $99 \cdot 2=198$ **(B)** such numbers.

17. Denote O_1, O_2, and O_3 as the centers of the bottom, middle, and top circles respectively, and let the bottom circle have radius r. Let Q_3 be the projection from O_3 to the line containing O_2 perpendicular to the tangent line, and Q_2 be the projection of O_2 to the line containing O_1 perpendicular to the tangent line. Then $O_2Q_3=4+1-3=2$ and $O_1Q_2=r-5$. Since $\triangle O_3Q_3O_2 \sim \triangle O_2Q_2O_1$,

$$\frac{r-3-2}{r+4}=\frac{2}{3+4}$$

from which $r=\frac{43}{5}$ **(C)**.

18. The numbers $1,2,\ldots,9$ are each replaced with two digits, contributing to a net gain of $9 \cdot 1=9$ digits in the new sequence. For any number $n>9$, there is a net gain of 1 digit if $n-\lfloor\sqrt{n}\rfloor^2 \geq 10$. Otherwise, the net gain is 0 as the two digit number is replaced with two single digit numbers. The satisfactory n are 35, and all positive integers in the intervals $[46,48],[59,63],[74,80],[91,99]$, giving a net gain of 25 digits. In summary, there are $9+25=34$ **(D)** more digits in the new sequence.

19. The number of digits in 2^n plus the number of digits in 5^n is equal to the number of digits in 10^n. Let $2^n=a \cdot 10^b$ and $5^n=c \cdot 10^d$, for $1<a,c<10$. It follows that $2^n \cdot 5^n=ac \cdot 10^{b+d}=10^n$, so $ac=10$ and $b+d=n-1$. We have $\frac{n}{3}-3+\frac{5n}{7}-1=n+1 \implies n=105$ **(D)**.

20. The numbers (a, b, c) must be pairwise relatively prime, so each of the numbers has at least two zeroes in the exponents of the prime factorization. The number 1200 can be prime factorized at $2^4 \cdot 3 \cdot 5^2$, and there are $3v_p(1200)$ ways to choose two exponents to be zero for $p \in \{2, 3, 5\}$ where $v_p(n)$ denotes the highest power of a prime p dividing n. We must also add 1 to each possiblity as all three exponents can be zero. Thus there are a total of $(3 \cdot 4 + 1)(3 \cdot 1 + 1)(3 \cdot 2 + 1) = 364$ **(E)** triples of (a, b, c).

21. Let O denote the circumcenter of $\triangle ADC$, and let $\angle B = x$. Then $\angle AOC = 180° - \angle B = 180° - x$. We know that $\angle ADB = 180° - \angle ADC = \frac{\angle AOC}{2} = 90° - \frac{x}{2}$ so triangle BAD is isosceles with $BA = BD$. Thus $BD = 5$ and $BC = 11$, so we can write $[ABC] = \frac{11}{5}[ABD] = \frac{11}{5} \cdot 12 = \frac{132}{5}$ **(C)**.

22. Let n be the missing value in the box. Note that $3n - 13 > 5m + n$ rearranges to $-n - 13 > 2m$ while $5m + n > 2m - 32$ rearranges to $3m > -n - 32$. Thus, we have

$$-3n - 39 > 6m > -2n - 64$$

$-3n - 39 > -2n - 64$ implies that $n < 25$. Plugging in $n = 25$ yields $-3n - 29 = -2n - 64 = -114$. When n decreases by one, $-3n - 29$ decreases by three while $-2n - 64$ decreases by two. We desire the values of n that trap only one multiple of six. Notice that there are $24 - n$ values of $n \in (-3n - 39, -2n - 64)$. By Pigeonhole Principle, any $n \leq 12$ fails, as there will always be two multiples of six out of twelve given numbers. Similarly, all $n \leq 18$ contain more than one multiple of six. We split into two cases:

- $19 \leq n \leq 24$. All intervals here fail besides $(-104, -99)$.
- $13 \leq n \leq 18$. All intervals here are valid, except for $(-81, -92)$, which contains two multiples of six.

There are 6 desired solutions, so our answer is **(C)**.

23. The parabola $y = ax^2 + bx + c$ passing through $y = \frac{2}{3}x$ at two lattice points implies that $ax^2 + \left(b - \frac{2}{3}\right)x + c$ can be factored as $a(x - k_1)(x - k_2)$ for distinct integers k_1 and k_2. Additionally, since the original parabola passes through $(3, 2021)$, the parabola $y = ax^2 + (b - \frac{2}{3})x + c$ must pass through $(3, 2019)$. Therefore we must maximize a for $a(k_1 - 3)(k_2 - 3) = 2019$. This is optimized through $(k_1, k_2) = (1, 2)$, giving $a = \frac{2019}{2}$ **(C)**.

24. The winning condition is equivalent to getting 11 or more out of $6 \cdot 4 \cdot 2 = 48$ equally likely possibilities.

   If Beth gets $1 - 5$ as her first draw, Arjun gets 6 on the next draw. One of Beth's 1st and 2nd draws must be 5, otherwise Arjun instantly wins on his second draw. If Beth gets $(5, 4)$ or $(4, 5)$ on first two, then she must get 2 on third draw. This gives $2 \cdot 1 = 2$ possibilities. If Beth gets $(5, 3)$ or $(3, 5)$ on first two, then she always loses. Similarly, she always loses for the other subcases.

   If Beth gets 6 on first draw, Arjun gets 5 next draw. Then, if Beth gets $1 - 3$ on her second draw, Arjun gets 4 next draw. In order for Arjun to lose, Arjun's 3rd draw must be 1. Then, Beth's second and third draws must be $(2, 3)$. This gives 2 possibilities. If Beth gets 4 on her second draw, Arjun gets 3 next draw. Notice now that Beth always win because her third draw is at least 1. This also gives 2 possibilities.

Therefore, the probability Beth wins is $\frac{6}{48} = \frac{1}{8}$ **(C)**.

25. Rearranging gives $y - x \in \left(\left(\frac{22-3\lfloor x\rfloor}{4}\right)^2, \left(\frac{23-3\lfloor x\rfloor}{4}\right)^2\right)$, so our desired region consists of eight parallelograms based on the value of $\lfloor x\rfloor$. For each $\lfloor x\rfloor$, the parallelogram has a side of length $\left(\frac{23-3\lfloor x\rfloor}{4}\right)^2 - \left(\frac{22-3\lfloor x\rfloor}{4}\right)^2 = \frac{45-6\lfloor x\rfloor}{16}$ and a height of length 1. Note that $0 \leq \lfloor x\rfloor < 8$, so our answer is
$$\sum_{\lfloor x\rfloor=0}^{7} \frac{45-6\lfloor x\rfloor}{16} = 12 \textbf{ (C)}.$$