1. Note that $9! = \frac{1}{10} \cdot 10!$, so $1 - \frac{1}{10} = \frac{9}{10} = 90\%$ **(E)** of the people remain in the city.

2. Suppose there are initially x red blocks and x blue blocks. There must be $8x$ red blocks added for red blocks to occupy $90\% = \frac{8x+x}{10x}$ of the total mass. The ratio of the mass of all red blocks to the original pile's mass is thus $\frac{8x}{2x} = 4 : 1$ **(C)**.

3. Choices (A),(B), and (E) can be disproven by letting $a = 1$. Choice (C) is false: decreasing a, despite b's relation to c, always decreases $a^{\frac{b}{c}}$. Choice **(D)** is true: increasing c decreases $\frac{b}{c}$ and thus increases $a^{\frac{b}{c}}$ where $a < 1$.

4. The longest diagonal of a regular hexagon are the three diameters of the hexagon's circumcircle that span across the hexagon's vertices. Let s be the length of the hexagon. Then, by dropping the altitudes from two adjacent vertices to the diagonal, we find that the diagonal has length $\frac{s}{2} + \frac{s}{2} + s = 2s$. The equilateral triangle with $2s$ as a length has area $4\sqrt{3}$. By the equilateral triangle area formula, $\frac{(2s)^2\sqrt{3}}{4} = \frac{4s^2\sqrt{3}}{4} = 4\sqrt{3} \implies s = 2$. Thus the perimeter is $6s = 6 \cdot 2 = 12$ **(C)**.

5. The first, second, and third conditions can be achieved by labeling the vertices in the order $(1, 2, 3, 4)$. The fourth condition implies that $1, 2$, and 4 are never adjacent; this is impossible as 2 must always be adjacent to either 1 or 4. Thus 3 **(D)** conditions have possible labelings.

6. The string ABA must be followed by BAB. Suppose that we use the labels $1 = ABAB, 2 = ABA, 3 = BAB, 4 = AB$. By casework on starting block, the possibilities are

$$1234, 1432, 2341, 2314, 4123, 4213.$$

   There are 6 **(C)** possibilities.

7. Note that AB is parallel to CD. Let $P = AC \cap BD$, then we have $\triangle CPD \sim \triangle BPA$. By the Angle Bisector Theorem, $CP = \frac{3}{2}$ and $AP = \frac{5}{2}$. Using similar triangles and the Pythagorean Theorem, $PD = \frac{3\sqrt{5}}{2}$ and $PB = \frac{5\sqrt{5}}{2}$, so $BD = PD + PB = 4\sqrt{5}$ **(D)**.

8. Each article of clothing can either be included or not included in the outfit, so there are a total of $(3 + 1)(4 + 1)(5 + 1) = 120$ **(D)** possible outfits.

9. Note that in each trapezoid $Q_nP_nP_{n+1}Q_{n+1}$, the shaded area is twice the unshaded area since the shaded area and unshaded area share the same height but the shaded area's base is twice the length of the that of the unshaded area, so the total unshaded area is $96 \cdot \frac{1}{3} = 32$ **(C)**.

10. $64^{\log_{64}(9)} = 9$, so we set $\log_{64} 9 = \log_{16} x$. It follows that $x = 9^{\log_{64} 16} = 9^{\frac{2}{3}} = \sqrt[3]{81}$ **(D)**.

11. Note that the digits must be some permutation of the remainders $\{1, 1, 0\}$ or $\{2, 2, 0\}$ when divided by three: 1 and 2 cannot be two of the three remainders and all three remainders cannot be uniform. If the remainders are $\{1, 1, 0\}$ and the zero is not in the front, there are $2 \cdot 3 \cdot 3 \cdot 4 = 72$ possiblities. If zero is in the front, we discard the zero digit and there are $3 \cdot 3 \cdot 3 = 27$ possibilities. So there are $72 + 27 = 99$ possibilities here. Likewise, the case for $\{2, 2, 0\}$ also has 99 possiblities. In total, there are $99 \cdot 2 = 198$ **(B)** such numbers.

12. Because EG is parallel to AF, $\triangle EBG \sim \triangle ABC$. Let $AE = x$ be the length of the rhombus. Then $EB = 14 - x$ and $\frac{EB}{EG} = \frac{AB}{BC} \implies \frac{14-x}{x} = \frac{14}{35} \implies x = 10$ **(D)**.

13. The number of digits in 2^n plus the number of digits in 5^n is equal to the number of digits in 10^n, since $\lceil\log_{10} 2^n\rceil + \lceil\log_{10} 5^n\rceil = \log_{10} 10^n + 1$. (Alternatively, let $2^n = a \cdot 10^b$ and $5^n = c \cdot 10^d$, for $1 < a, c < 10$, and observe what happens for the product of 2^n and 5^n). We have $\frac{n}{3} - 3 + \frac{5n}{7} - 1 = n + 1 \implies n = 105$ **(D)**.

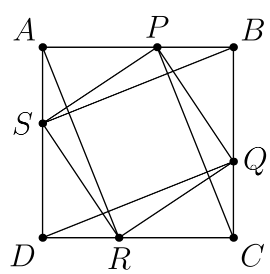
14. The inequality rearranges to $\sin^2(x+y) < 1 - \cos^2(x) = \sin^2(x) \implies \sin(x+y) - \sin(x) < 0$. We change $\sin(x+y) - \sin(x)$ to equal $\sin(x)\cos(y) + \cos(x)\sin(y) - \sin(x) = 2\sin(\frac{y}{2})\cos(\frac{2x+y}{2})$ by the converse of the sine-cosine multiplication rule. Note that we must have $\cos(\frac{2x+y}{2}) < 0$ as the sine part is always nonnegative, so $y - 2x > \pi$. The line $y - 2x > \pi$ divides the square region of all posiible points (x, y) into two regions, with the desired region being a triangle whose base is a fourth of the square's upper side. The answer is thus $\frac{1}{4}$ **(A)**.

15. The numbers $1, 2, \ldots, 9$ are each replaced with two digits, contributing to a net gain of $9 \cdot 1 = 9$ digits in the new sequence. For any number $n > 9$, there is a net gain of 1 digit if $n - \lfloor\sqrt{n}\rfloor^2 \geq 10$. Otherwise, the net gain is 0 as the two digit number is replaced with two single digit numbers. The satisfactory n are 35, and all positive integers in the intervals $[46, 48], [59, 63], [74, 80], [91, 99]$, giving a net gain of 25 digits. In summary, there are $9 + 25 = 34$ **(D)** more digits in the new sequence.

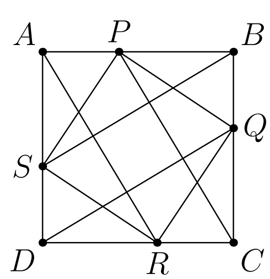
16. The numbers (a, b, c) must be pairwise relatively prime, so each of the numbers has at least two zeroes in the exponents of the prime factorization. The number 1200 can be prime factorized at $2^4 \cdot 3 \cdot 5^2$, and there are $3v_p(1200)$ ways to choose two exponents to be zero for $p \in \{2, 3, 5\}$ where $v_p(n)$ denotes the highest power of a prime p dividing n. We must also add 1 to each possiblity as all three exponents can be zero. Thus there are a total of $(3 \cdot 4 + 1)(3 \cdot 1 + 1)(3 \cdot 2 + 1) = 364$ **(E)** triples of (a, b, c).

17. Note that the vertices of the smaller square must split the sides of the larger square into lengths 2 and 3, since $\sqrt{2^2 + 3^2} = \sqrt{13}$. We have two cases to consider:

Case 1: $AP = 3$ gives the following configuration.



Case 2: $AP = 2$ gives the following configuration.



$AP = 3$ gives the quadrilateral with more area. To compute the area of the region, we subtract the area of the square from the areas of $\triangle ABS, \triangle BPC, \triangle QCD$, and $\triangle ARD$. Then we add back the four pairwise intersections between these four triangles. Our answer is computed to be $25 - 4 \cdot 5 + 4 \cdot \frac{20}{29} = \frac{225}{29}$ **(D)**.

18. If $z^3 = 1$ and $z \neq 1$, then $z^2 + z + 1 = 0 \implies z = -z^2 - 1$. Then

$$\sum_{n=1}^{9} \frac{z + n^2 + 1}{z - n} = \sum_{n=1}^{9} \frac{-z^2 - 1 + n^2 + 1}{z - n} = \sum_{n=1}^{9} -z - n$$

The real part of z is $-\frac{1}{2}$, so thus the real part of the sum is $9 \cdot \frac{1}{2} - (1+2+\cdots+9) = \frac{-81}{2}$ **(A)**.

19. Let $P(x) = x^2 + bx + c$. Write $P(P(x)) = (x^2 + bx + c)^2 + b(x^2 + bx + c) + c$. Then $P(P(x)) - 12 \cdot P(x) + 20 = 0$ expands to a quartic with leading coefficient 1 and constant term $c^2 + bc + c - 12c + 20$. By Vieta's $c^2 + bc + c - 12c + 20 = -4$ or $c(c + b - 11) = -24$. $c(11 - b - c) = 24$ implies $11 - b - c$ must be a positive integer. Since $c + (11 - b - c) < 11$, we must have $c = 4$ and $11 - b - c = 6$ or $c = 6$ and $11 - b - c = 4$. In both cases $b = 1$, so $P(x) = x^2 + x + 4$ or $P(x) = x^2 + x + 6$. The sum of all possible values of $P(1)$ is $6 + 8 = 14$ **(D)**.

20. Let M and N be the midpoints of AB and CD, respectively. Clearly M and N are the circumcenters of $\triangle ABE$ and $\triangle CDE$. By angle chase,

$$\angle ABF = \angle AEF = 180^\circ - \angle CEF = \angle CDF = 30^\circ,$$

implying $\angle AMF = \angle CNF = 60^\circ$. Thus, because $AM = FM$ and $CN = FN$, we must have that $\triangle AMF$ and $\triangle CNF$ are equilateral. Additionally, $\angle AFC = \angle AFM + \angle MFC = \angle CFN + \angle MFC = \angle MFN$. We have $\triangle AFC \cong \triangle MFN$. Since, $AFB = \angle AEB = \angle CED = \angle CFD$, it follows that $\angle AFC = \angle AFB + \angle BFC = \angle CFD + \angle BFC = \angle BFD$ and $\triangle AFC \sim \triangle BFD$. Now $\tan 30^\circ = \frac{AF}{BF} = \frac{AC}{BD} = \frac{4}{BD}$, so $BD = 4\sqrt{3}$. As such, the area of $ABCD$ is $\frac{AC \cdot BD}{2} = \frac{4 \cdot 4\sqrt{3}}{2} = 8\sqrt{3}$ **(E)**.

21. Let n be the missing value in the box. Note that $3n - 13 > 5m + n$ rearranges to $-n - 13 > 2m$ while $5m + n > 2m - 32$ rearranges to $3m > -n - 32$. Thus, we have

$$-3n - 39 > 6m > -2n - 64$$

$-3n - 39 > -2n - 64$ implies that $n < 25$. Plugging in $n = 25$ yields $-3n - 29 = -2n - 64 = -114$. When n decreases by one, $-3n - 29$ decreases by three while $-2n - 64$ decreases by two. We desire the values of n that trap only one multiple of six. Notice that there are $24 - n$ values of $n \in (-3n - 39, -2n - 64)$. By Pigeonhole Principle, any $n \leq 12$ fails, as there will always be two multiples of six out of twelve given numbers. Similarly, all $n \leq 18$ contain more than one multiple of six. We split into two cases:

- $19 \leq n \leq 24$. All intervals here fail besides $(-104, -99)$.
- $13 \leq n \leq 18$. All intervals here are valid, except for $(-81, -92)$, which contains two multiples of six.

There are 6 desired solutions **(C)**.

22. The winning condition is equivalent to getting 11 or more out of $6 \cdot 4 \cdot 2 = 48$ equally likely possibilities.

If Beth gets $1 - 5$ as her first draw, Arjun gets 6 on the next draw. One of Beth's 1st and 2nd draws must be 5, otherwise Arjun instantly wins on his second draw. If Beth gets $(5, 4)$ or $(4, 5)$ on first two, then she must get 2 on third draw. This gives $2 \cdot 1 = 2$ possibilities. If Beth gets $(5, 3)$ or $(3, 5)$ on first two, then she always loses. Similarly, she always loses for the other subcases.

If Beth gets 6 on first draw, Arjun gets 5 next draw. Then, if Beth gets $1 - 3$ on her second draw, Arjun gets 4 next draw. In order for Arjun to lose, Arjun's 3rd draw must be 1. Then, Beth's second and third draws must be $(2, 3)$. This gives 2 possibilities. If Beth gets 4 on her second draw, Arjun gets 3 next draw. Notice now that Beth always win because her third draw is at least 1. This also gives 2 possibilities.

Therefore, the probability Beth wins is $\frac{6}{48} = \frac{1}{8}$ **(C)**.

23. Let E be the midpoint of AC and F be the point on BC such that $CF = CP$. For the sake of clarity, we let P denote only the starting position of the moth, the midpoint of AB.

It is clearly impossible for the moth to increase its distance from M, so it cannot go outside the circle with center M and radius PM. Similarly, it is impossible for the moth to go outside the circle with center C and radius PC. This eliminates the region on the same side of A as arc $\widehat{PE}$ (with center M) and the region on the same side of B as arc $\widehat{PF}$ (with center C). We now verify the validity of certain regions. The region bounded by line segments PM and FM and arc $\widehat{PF}$ is achievable because the moth can go radially inward towards M from any point on the arc and the distance to C will decrease as well.

The region bounded by line segments PM and EM and arc $\widehat{PE}$ (which is a circular sector) is achievable because the moth can go radially inward from P towards M for some distance and rotate in a circle centered at M in the direction of segment EM. It's not hard to see why the second movement decreases distance from C at all times.

It remains to check points in triangle EMC. It appears that going radially inward from P towards M for some distance and rotating in a circle centered at M towards segment CM would cover this area, and algebraically, this does hold. However, the moth cannot exit the triangle. If we let X denote the midpoint of EC, then segment EX blocks some hypothetical circular paths. What we have is that while triangle MEX is unaffected by this phenomenon, triangle CMX is. The points S in triangle CMX that are further from M than X (on the same side of $\widehat{XG}$ as C, where G is on MC with $MG = MX$) are not achievable, because to reach S, the moth must pass through a point on segment MX first, so after that time, the moth cannot increase its distance from M. The rest of the points in triangle CMX are unaffected by the segment EX block. Therefore, the desired region is the entire triangle except for three regions, one located near each of the three vertices.

$$[\widehat{PFC}] + [\widehat{PEC}] - [\triangle MXC] + [\widehat{XGM}]$$

$$= [\widehat{PFC}] + [\widehat{PEM}] - [\triangle MXC] + [\widehat{XGM}]$$

$$= \frac{(\frac{\sqrt{3}}{2})^2\pi}{12} + \frac{(\frac{1}{2})^2\pi}{6} - \frac{\frac{\sqrt{3}}{4} \cdot \frac{1}{4}}{2} + \frac{(\frac{\sqrt{3}}{4})^2\pi}{12}$$

$$= \frac{23\pi - 6\sqrt{3}}{192}$$

yielding 221 **(E)**.

24. Rearranging gives $y - x \in ((\frac{22-3\lfloor x \rfloor}{4})^2, (\frac{23-3\lfloor x \rfloor}{4})^2)$, and our desired region consists of eight parallelograms based on the value of $\lfloor x \rfloor$. For each $\lfloor x \rfloor$, the parallelogram has a side of length $(\frac{23-3\lfloor x \rfloor}{4})^2 - (\frac{22-3\lfloor x \rfloor}{4})^2 = \frac{45-6\lfloor x \rfloor}{16}$ and a height of length 1. Note that $0 \leq \lfloor x \rfloor < 8$, so our answer is
$$\sum_{\lfloor x \rfloor = 0}^{7} \frac{45 - 6\lfloor x \rfloor}{16} = 12 \textbf{ (C)}.$$

25. Let O_1 and O_2 be the centers of ω_1 and ω_2, respectively. Extend DP to meet ω_2 again at D'. The negative homothety between the two circles implies $PD' = 2$, so $\mathrm{Pow}(D, \omega_2) = DP \cdot DD' = 1 \cdot 3 = 3 = O_2D^2 - 4^2$, and $O_2D = \sqrt{19}$. Let the tangent points of D to ω_2 be E and F. Now, notice that due to O_2EDF being cyclic, $\angle XDY = \angle EO_2F = 2\angle EO_2D$. Also note that $2r_1 \sin XDY = 4 \sin XDY = XY$, so it suffices to compute $\sin\angle XDY = \sin(2\angle EO_2D)$. In right triangle O_2ED, we have $O_2D = \sqrt{19}$, $OE = 4$, so $DE = \sqrt{3}$. Thus $\sin(2\angle EO_2D) = 2\sin\angle EO_2D \cos\angle EO_2D = 2 \cdot \frac{\sqrt{3}}{\sqrt{19}} \cdot \frac{4}{\sqrt{19}} = \frac{8\sqrt{3}}{19}$. $XY = \frac{32\sqrt{3}}{19}$, giving 54 **(C)**.